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BUREAU OF STANDARDS
S. W. STRATTON, Director

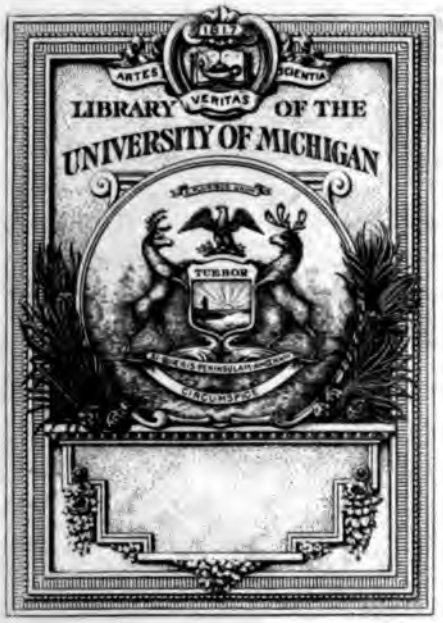
A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY

BY
EDWARD B. ROSA, Physicist
and
N. E. DORSEY, Associate Physicist
Bureau of Standards

[MAY 20, 1907]

REPRINT NO. 65
(FROM BULLETIN OF THE BUREAU OF STANDARDS, VOL. 3, NOS. 3 AND 4)

WASHINGTON
GOVERNMENT PRINTING OFFICE
1907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A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

By E. B. Rosa and N. E. Dorsey.

"Etant donné l'intérêt qui s'attache à la détermination de la *vitesse v*, il paraît désirable que de nouvelles expériences soient entreprises. La précision des anciennes mesures peut être dépassée: *toutes les méthodes s'y prêtent.* Il y a encore à réduire quelques corrections trop incertaines; il y a à simplifier quelques mesures auxiliaires trop complexes, et par ce nouvel effort on pourra, sans aucun doute, apporter dans la mesure de *v* une précision supérieure à celle aujourd'hui acquise pour la vitesse de la lumière." (H. Abraham, Rapport présenté au Congrès international de Physique, Paris, 1900.)

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I. INTRODUCTION.

1. THE METHOD.

Although many determinations of this important quantity have been made, even the best results hitherto found differ considerably from one another, and none are sufficiently accurate to be entirely satisfactory. The earlier results were rough, due largely to imperfect apparatus. Some of the more recent determinations have been made by methods that are interesting and important in themselves, but not capable of yielding results of high accuracy. Few, if any, even of the best determinations are exact enough to justify a claim of definitely fixing the value of v to within one part in a thousand. Abraham, in his report to the Paris Congress of 1900, expresses a doubt whether as a result of all the determinations that have been made we can be sure of the value of v to within 1 part in 1,000.

One of us published[1] a determination of v in 1889, made at the Johns Hopkins University, using the spherical condenser which had been employed by Rowland ten years before, deriving v from the ratio of the electrostatic to the electromagnetic capacity of the condenser. This is probably the best of all the methods yet employed for determining the ratio of the units. By this method v is determined in terms of a resistance, but inasmuch as v is equal to the square root of the ratio of the capacities, the uncertainty due to an error in the value of the ohm is divided by two, so that if we admit an uncertainty of two parts in five thousand in the value of the international ohm, that would involve an uncertainty of only one part in five thousand in v. All of the methods of determining v which do not involve the ohm are subject to even larger uncertainties in other directions. Hence it seemed to us desirable to make a new determination of the value of v, using the method of capacities, and to undertake to attain a higher order of accuracy than had heretofore been realized.

[1] Phil. Mag., vol. 28, p. 315, 1889, and Am. Jour. Sci., vol. 38, p. 298, 1889. In this article I expressed the hope of repeating the determination of v the following year in order to ascertain the cause of certain systematic differences appearing in the results. A favorable opportunity, however, did not arise until fifteen years afterwards, when the superior facilities of the Bureau of Standards suggested that the time had arrived for a new and more accurate determination of v than was possible in the former work.—E. B. R.

2. SPHERICAL CONDENSERS.

Through the courtesy of Prof. Joseph S. Ames, of the Johns Hopkins University, we secured the same spherical condenser that had been employed in the experiments of 1879 and 1889, already referred to. The two balls were repolished and the two halves of the shell were reground so that they fitted together water-tight. The two electrostatic capacities of the condenser were redetermined, using first the larger ball and then the smaller ball within the shell. The electromagnetic capacity was found by means of the Wheatstone bridge, using a rotating commutator. The experiments were begun early in the summer of 1904, but owing to the difficulty of obtaining sufficiently high insulation at that time they were deferred until the following winter. Meanwhile we designed a new rotating commutator better fitted for this experiment than the one used in the preliminary work and had it constructed in the instrument shop of the Bureau.

On resuming the work in the winter of 1904–5, we were able to obtain much better measurements of the electromagnetic capacity than before, but we were unable to get a sufficiently exact determination of the capacity of the charging wire which passed through the small hole at the pole of the outer hollow sphere, touching the inner sphere. When this wire is lifted far enough to break the contact with the ball its capacity changes for two separate reasons—first, because its length within the shell has changed, and, second, because its potential is then not the same as that of the ball; and hence the field is different after the contact is broken. This difficulty we afterwards overcame completely by getting the capacity with two charging wires, one at the pole and one at the equator. First withdrawing the polar charging wire, the difference in electromagnetic capacity gave the capacity of the charging wire itself, the ball meanwhile being charged by a wire at the equator. Then replacing the polar charging wire and withdrawing that at the equator, the difference gave the capacity of the latter. This gives a determination of the correction for the capacity of the charging wire that is free from theoretical objection, and which in practice gave remarkably accurate results.

3. CYLINDRICAL CONDENSERS.

Before this difficulty due to the charging wire was overcome, however, we designed and had constructed in the instrument shop of the Bureau (early in 1905) a set of cylindrical condensers[1] which could be measured without the necessity of determining the capacity of the charging wire, in the way since described by Lord Rayleigh.[2] A pair of coaxial cylinders was mounted on a suitable base with their axes vertical, and so joined to the Maxwell bridge that the electromagnetic capacity C_1 of the inner cylinder could be measured. A second pair of cylinders of the same radii was then added to the first and the increased capacity C_2 determined. Then a third pair was added and the total capacity C_3 determined. The difference C_2-C_1 is then the capacity of the second pair of cylinders, while C_3-C_2 is the capacity of the third pair, assuming that the unknown end corrections and the capacity of the charging wire (which has remained undisturbed) have not changed during the process of building up the cylinders by adding successive sections.

4. GUARD CYLINDERS.

This method gave as good results as we had anticipated, but in the meantime we had so much improved the work with the spherical condensers that the results with the cylinders were not satisfactory. We then added a fourth pair of cylinders to the set and insulated the two end sections of the inner cylinders so that they formed guard cylinders, after the manner of the guard rings of a plate condenser. The upper guard cylinder was provided with a micrometer screw, so that the breadth of the air gap could be varied and the computed correction for the gap checked by experimental determinations with varying widths. The use of the guard cylinders necessitated a modification of the rotating commutator for charging and discharging the condensers, by adding three brushes on the opposite side of the ring, so that the middle cylinders and the guard cylinders could be charged and discharged simultaneously. Using the guard cylinders in this way we obtained results of much greater uniformity and greater apparent accuracy than we had expected.

[1] We are under obligations to Mr. H. B. Brooks for assistance in designing this condenser. It and some other apparatus employed was built by Mr. Joseph Ludewig in the instrument shop of this Bureau.

[2] Phil. Mag., 12, p. 97; 1906.

5. DIFFERENTIAL METHOD.

We therefore thought it wise to check the work done by the Maxwell bridge method by independent measurements, using a differential method and a differential galvanometer. The Weston Electrical Instrument Company built us for this work a very delicate differential moving coil galvanometer, with a bifilar suspension (both above and below) of fine silver wire. As this instrument could be used not only in the differential method but also in the bridge method (with the two coils in series), we were able to compare the results by the two methods very critically, as we could change over from one to the other in a moment, using the same resistances. The results agreed with great exactness. In the differential method we measured sometimes the charge and sometimes the discharge. No difference whatever could be observed.

We had now obtained a considerable number of determinations of the ratio **v**, using two different spherical condensers (the same shell in each case, but two balls of different sizes), two different capacities of cylindrical condensers, both with and without guard cylinders, and had made the measurements (both on spheres and cylinders) by two very different means, namely, the Maxwell bridge and the differential galvanometer methods. We had, moreover, varied the conditions in many ways, such as changing the speed of the commutator and so the frequency of the charge and discharge; varying the voltage of the battery; interchanging resistances; varying the galvanometers and other accessory apparatus; measuring both the charge and the discharge; and working in both summer and winter, at higher and lower temperatures (always of course correcting results to a standard temperature of 20°) and with considerable changes in atmospheric humidity. In summer this was kept low enough to get sufficiently high insulation by the use of coils of iron pipe in the room, through which cold calcium chloride brine was circulated, which condensed the surplus moisture from the atmosphere and enabled us to maintain a nearly constant temperature and a moderate humidity. In the exceptional cases when this could not be done observations were discontinued.

6. PLATE CONDENSERS.

In order to make the series of measurements more complete and to have one more chance of detecting any constant error that might possibly still remain in the mean value of v which we were obtaining, we designed and had built in the instrument shop of the Bureau a parallel plate, guard ring condenser.[4] This is not as favorable a form of condenser as the others, and possesses some disadvantages that we did not appreciate when it was designed. We have, however, obtained fair results, which confirm the value for v given by the other condensers, although the values found by the plate condenser are entitled to less weight than those obtained by the spherical condensers and the cylindrical condensers with end guard cylinders.

The value which we have found for the ratio of the electrostatic to the electromagnetic unit, resulting from all our work (carried on without interruption since November, 1904) is

$$v = 2.9963 \times 10^{10}$$

taking the dielectric constant of air as unity. Referring to vacuum this becomes

$$v = 2.9971 \times 10^{10}$$

We believe this result is correct to within 1 part in 10,000, barring any uncertainty in the value of the international ohm. This is more fully discussed below.

II. DESCRIPTION OF THE CONDENSERS.

7. SPHERICAL CONDENSER.

As already stated, the spherical condenser was loaned to the Bureau by the Johns Hopkins University. It was made in 1879 from the designs and specifications of the late Prof. H. A. Rowland. The construction of the condenser is shown in Figs. 1 and 2. It consists of a brass shell provided with leveling screws and divided along its equator so that the upper hemisphere can be removed. The interior of the shell is accurately worked to a spherical surface

[4] We are under obligations to Mr. F. S. Durston for assistance in designing this condenser and the direct reading chronograph described below. The plate condenser was built by Mr. Oscar Lange, chief mechanician of this Bureau.

and is nickel-plated. At the pole of the upper hemisphere is a hole through which passes one of the charging wires and also the silk

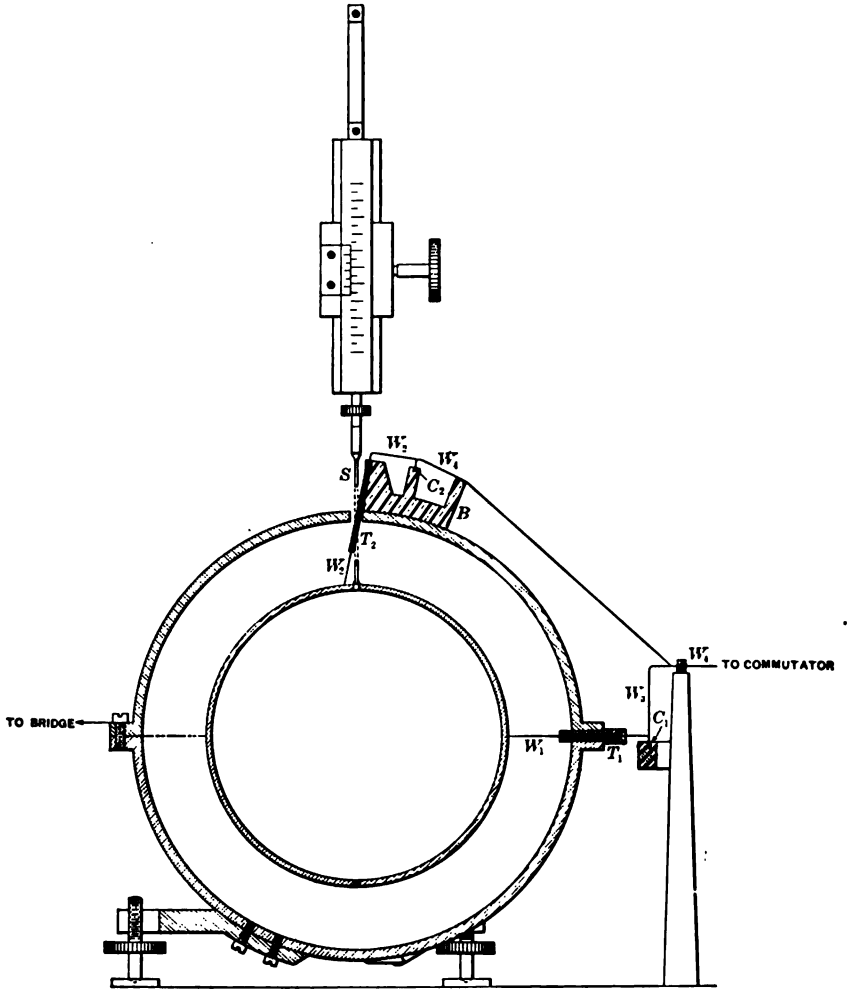

Fig. 2.—*Section of Spherical Condenser.*

The inner sphere is suspended by the silk cord S at the center of the shell. The two charging wires W_1 and W_2 are guided by the fixed tubes T_1 and T_2. The outside ends of W_1 and W_2 dip into the small mercury cups C_1 and C_2. These mercury cups were joined by the wires W_3 and W_4 to the rotating commutator.

cord by which the inner ball is suspended. In the equatorial plane of the shell is a second hole, through which we passed the second

charging wire. The inner ball is either one of two carefully spun nickel-plated balls of different radii. These with the shell form two condensers of different capacities. The silk cord supporting the ball, after passing through the hole in the upper hemisphere of the shell, is attached to the lower end of a scale which can be moved vertically in guides with a rack and pinion adjustment. Fastened to one of the guides is a vernier, by means of which the position of the scale can be directly read to a tenth of a millimeter. From the upper end of the scale a strip of thin spring steel passes to one end of a lever to the other end of which is attached a counterweight. The end of the lever to which the spring steel is attached is cut to the arc of a circle with its center at the pivot of the lever. The radius of the circle being equal to the distance of the plane of the scale from the pivot of the lever, the tension upon the scale will always be in the direction of its length whatever the position of the lever, within working range.

8. ADJUSTING THE CONDENSER.

In assembling the condenser the upper rim of the lower hemisphere is first carefully leveled. The ball and the upper hemisphere are then put in place and the ball adjusted by the rack and pinion movement until its center is approximately in the plane of the rim of the lower hemisphere. The upper hemisphere is now raised and propped on blocks so that the ball can be reached along the horizontal plane between the hemispheres, care being taken that the silk suspension of the ball hangs freely through the hole in the top of the hemisphere. The horizontal position of the ball is now tested at four points by means of a brass distance piece resting upon the broad rim of the lower hemisphere. Then by carefully sliding the lower hemisphere one way or another the horizontal centering of the ball can be attained with an error probably not greater than 0.1 mm. The contact between the ball and the distance piece is determined electrically.

This adjustment being completed the upper hemisphere is lowered and fastened in place on the lower one. Then by alternately raising and lowering the ball, by the rack and pinion adjustment, so as to make contact with the top or with the bottom of the shell, contact being determined electrically, it is easy to determine the scale read-

ing to $\frac{1}{10}$ mm corresponding to the central position of the ball. Setting the scale to this reading the condenser is ready for use.

This centering was later tested by measuring the capacity of the condenser in its adjusted position, and then displacing the ball first up and then down by known amounts. The capacity is a minimum with the ball in the center, and increases approximately as the square of the displacement in either direction. This gave a very delicate method of verifying the centering.

The warping of the floor and of the table, occasioned by variations in the humidity, causes the level of the instrument to change more or less. This of course displaces the ball from its central position slightly and so varies the capacity. In the early portion of the work the importance of this source of error was not fully appreciated, but later a delicate level was kept on the brass arm supporting the guides for the movable scale and the level of the instrument was readjusted as often as need be. The magnitude of the error that may be thus introduced is small but appreciable.

The radius of the shell is approximately 12.6~1 cm and of the balls 10.118 cm and 8.874 cm approximately.

9. THE CHARGING WIRES.

The method o. charging the condenser is of great importance and involves the determination of a relatively large correction term. The charging can be done only by means of a wire reaching the ball through a hole in the shell. This wire produces two distinct effects, both of which must be eliminated: (1) It has a relatively large capacity which is thus added to that of the ball; (2) its presence distorts the field of force within the shell. These effects can not be eliminated by any process involving the mere withdrawal in whole or in part of the wire, or by a lowering of the ball, but must be definitely measured and allowed for. Furthermore, the capacity of the wire is greatly affected by slight changes in its position with respect to the shell, especially with respect to the edges of the hole through which it passes. Hence we must rigidly fix the position of the charging wire and then accurately determine its capacity when in this position.

In order to accomplish this we have employed two charging wires (Fig. 2). One extended diagonally through the hole in the

top of the shell through which hung the silk thread supporting the ball; the second extended horizontally through a hole in the equatorial plane of the shell. In the latter hole was tightly fitted an ebonite bushing extending 12 mm outside the outer surface of the shell and 5 mm inside the inner surface. The hole through the center of this was wide at the two ends of the bushing, but for about 2 centimeters in its central portion was of such a size as just to allow a copper wire (1.3 mm in diameter) to slide with gentle friction. The enlarging of the hole at the ends of the bushing, as well as the extension of the bushing beyond the surfaces of the shell, were of course intended to improve the insulation of the charging wire. The charging wire used for this side hole consisted of the above-mentioned piece of copper wire, to the inner end of which were soldered a couple of very fine phosphor bronze wires 2 or 3 mm in length. To the outer end of the charging wire was soldered a section of fine copper wire (0.13 mm diameter) bent so that by rotating the charging wire when in position the end of this fine wire could be dipped into a minute mercury cup just large enough to receive it and a similar wire connecting permanently with the rest of the charging system. The phosphor bronze tips allow good contact to be made with the ball without appreciably displacing it from its undisturbed position. The position of the charging wire is fixed by a mark on the wire in the plane of the end of the ebonite bushing.

To the top of the shell was fastened with hard wax a block of ebonite carrying a slender ebonite tube, which extended through the hole in the shell, just clearing the cord supporting the ball and reaching within a few millimeters of the surface of the larger ball when the latter is in position. This tube was made long, on account of the necessity of using a much finer wire, than in the case of the hole in the side of the shell; the hole through this tube was of such a size that a copper wire (0.72 mm diameter) could just pass without danger of jamming. This wire also had fine phosphor bronze tips soldered to its lower end; its upper end was so bent that when the wire was pushed in the tube so that the phosphor bronze tips made good contact with the ball the bent portion was in contact with the top of the ebonite block. To the outer end of this also was soldered a section of fine copper wire of such a length as just to dip into a diminutive mercury cup when the charging wire was in place.

The two small mercury cups were permanently connected with the rest of the charging system by fine wires stretched taut between rigid supports, so as to prevent accidental variations in capacity. The position of the upper charging wire was thus uniquely determined, and its capacity could vary only by a slipping of the ebonite block waxed to the shell; this block did slowly slip as a result of the tension of the wire running from it to the rest of the apparatus, but the variation was not appreciable in a single day, and the capacity of the charging wire was redetermined every time the capacity of the condenser was measured. There was no chance for such a change in the capacity of the side charging wire.

Having definitely fixed the position of the charging wires the next thing was to measure their capacities. This was done by a method of differences. The side wire w_1 being in position and the top one w_2 removed, the electro-magnetic capacity K'_1 of the commutator, leads, and ball, was measured as accurately as possible. Then w_2 was placed in position, everything else remaining as before, and the total capacity K' was measured. The difference between these two capacities, $K'-K'_1$, consists of three terms: (1) The capacity of w_2; (2) the change in the capacity of the ball due to the redistribution of charge upon it, resulting from the presence of w_2; (3) the change in capacity of the mercury cup A and of the wire leading to it, produced by the presence of w_2. Call the sum of these three effects c_2, the effective capacity of w_2.

If we now remove w_1, leaving everything else unchanged and measure the resulting total capacity K'_2, the difference between this capacity and the capacity (K') with *both* w_1 and w_2 in position will give us c_1, the effective capacity of w_1. This is made up of three terms corresponding exactly with those composing the effective capacity of w_2.

If we now remove both w_1 and w_2 and measure the remaining capacity k, which is made up of the capacity of the commutator and of the wires leading to (and including) the mercury cups C_1 and C_2, then put w_1 in place, everything else remaining unchanged, and again measure the total resulting capacity K'_1, the difference of these two capacities, K'_1-k, will be exactly equal to the capacity of the ball increased by what we have called c_1 the effective capacity of w_1 and which we have already measured. Thus K'_1-k-c_1 is the electromag-

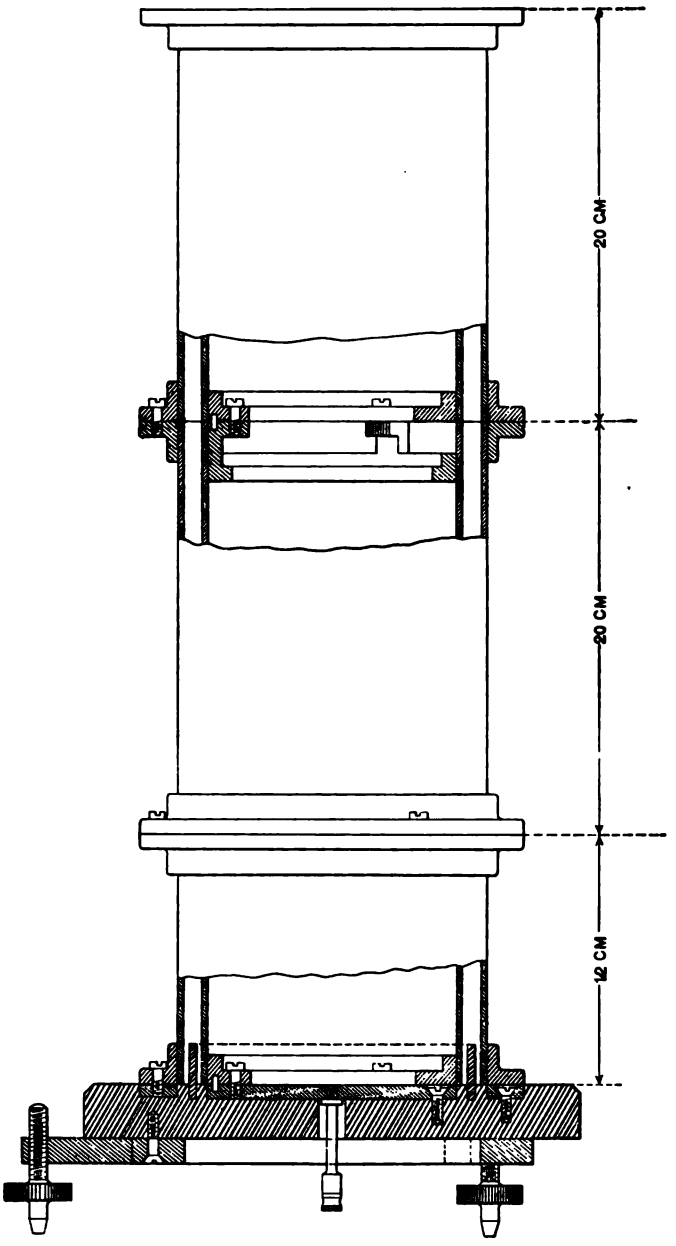

Fig. 3.—*Cylindrical Condenser of Three Sections. No Guard Cylinder.*

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netic capacity C_A, or C_B, of the ball *entirely free* from all effects produced by the charging wire.[5]

10. CYLINDRICAL CONDENSER.

As stated above, the cylindrical condenser was built in the instrument shop of the Bureau. The condenser, as first constructed, consisted of three sections—a base section 12 cm long and two other sections each 20 cm long. Later a fourth section, 12 cm long, was made to serve as an upper guard cylinder. The radii of the inner cylinders are, approximately, 6.257 cm, those of the outer ones 7.241 cm. The base of the instrument is a heavy ebonite block mounted on leveling screws. In the top of the block is set a circular brass plate of slightly smaller diameter than the inner cylinder. This plate, to which the lower section of the inner cylinder is attached by screws, entirely closes the bottom of the inner cylinder. In the ebonite base, midway between the two cylinders, is set an ebonite ring, 2 mm thick and 1 cm high, thus making the leakage path between the cylinders about three times as long as it would otherwise have been. Beyond this a brass ring of a slightly larger inner diameter than the outer cylinders is set in the ebonite, and the lower section of the outer cylinder is attached to this ring by screws. Connection to the outer cylinder is made by means of one of the screws fastening it to the base. Connection to the inner cylinder is made by means of a binding post set in the middle of the under side of the ebonite base and connecting with the plate which closes the bottom of the inner cylinder. The various sections of the cylinders are interchangeable and are fastened together by three screws and three steadying pins at each joint. A brass plate of the internal diameter of the inner cylinders, provided with steadying pins and

[5] It has been stated (Abraham, Rapport, Congrès International de Physique, 1900) that Rowland, who used this condenser in his determination of the ratio of the units in 1879, failed to eliminate the capacity of the charging wire. This is only partially correct. As we have seen, the charging wire produces two effects. The portion due to its own capacity was eliminated in Professor Rowland's work, for, after charging the condenser, the charging wire was withdrawn and a second wire was introduced to discharge it. (See p. 267, Physical Papers.) It was the quantity discharged that he measured electromagnetically. On the other hand, the effect of the distortion of the field by the wire was not eliminated. This quantity, however, was very small and need not be considered when an accuracy of not more than 1 in 1,000 is sought.

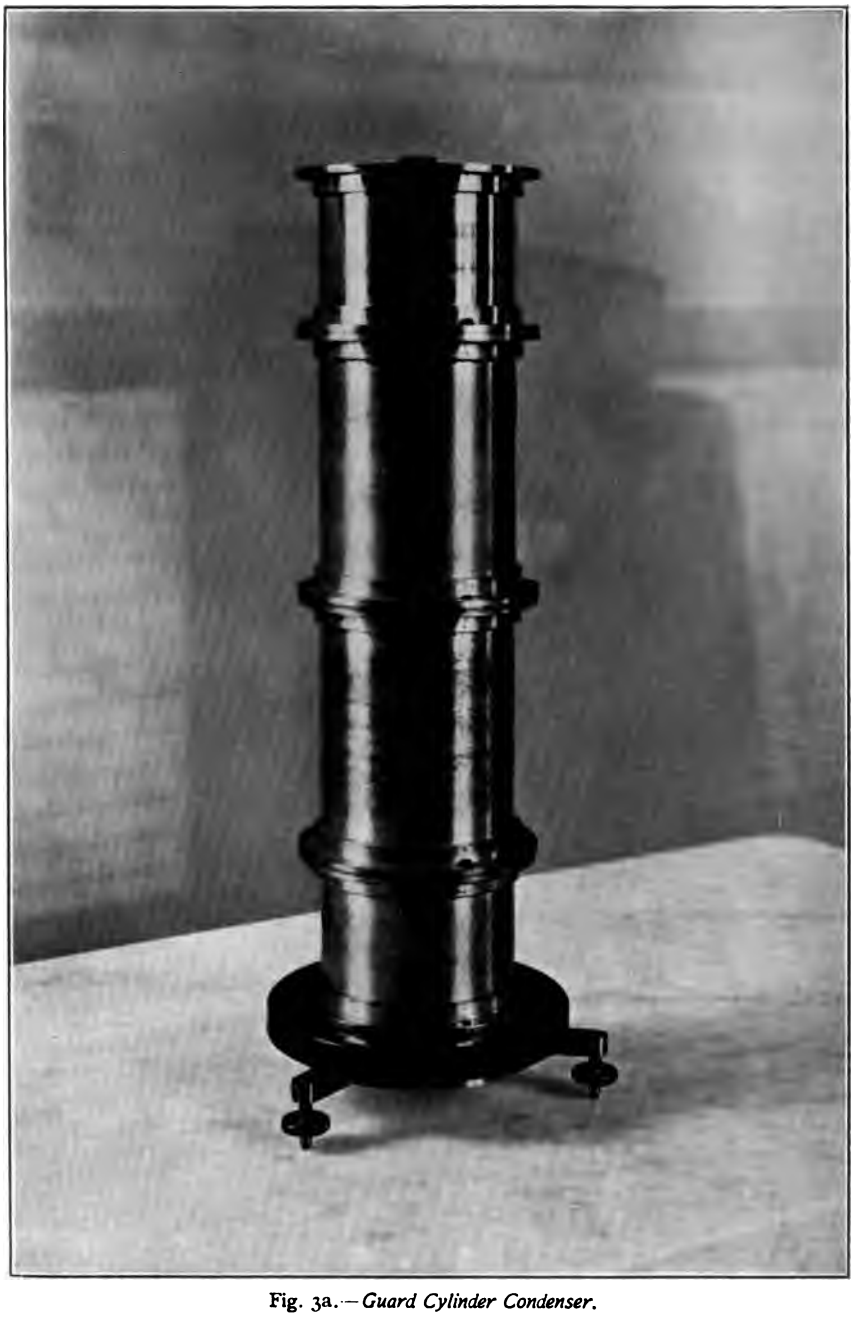

Fig. 3a.—*Guard Cylinder Condenser.*

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with short projections to rest upon the upper rim of the cylinder, is used for closing the upper end of the inner cylinder when desired.

The cylinders are of brass, silver plated both inside and out. Sections of tubing of the required diameter were cut to the proper

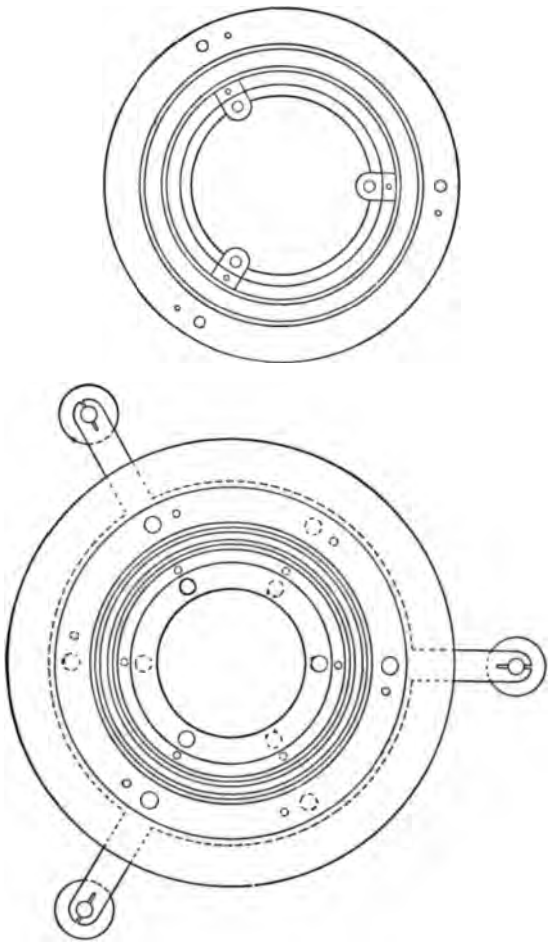

Fig. 4.—*Plan of Base of Cylindrical Condenser, showing Method of Attachment of Cylinders. Plan of End of a Section of the Condenser, showing Location of Screws and of Steadying Pins.*

lengths, and each section was then stiffened by having heavy rings soldered to it near each end, the rings being, of course, placed inside the inner cylinders and outside the outer ones; the screws and steadying pins for connecting the sections pass through the rings.

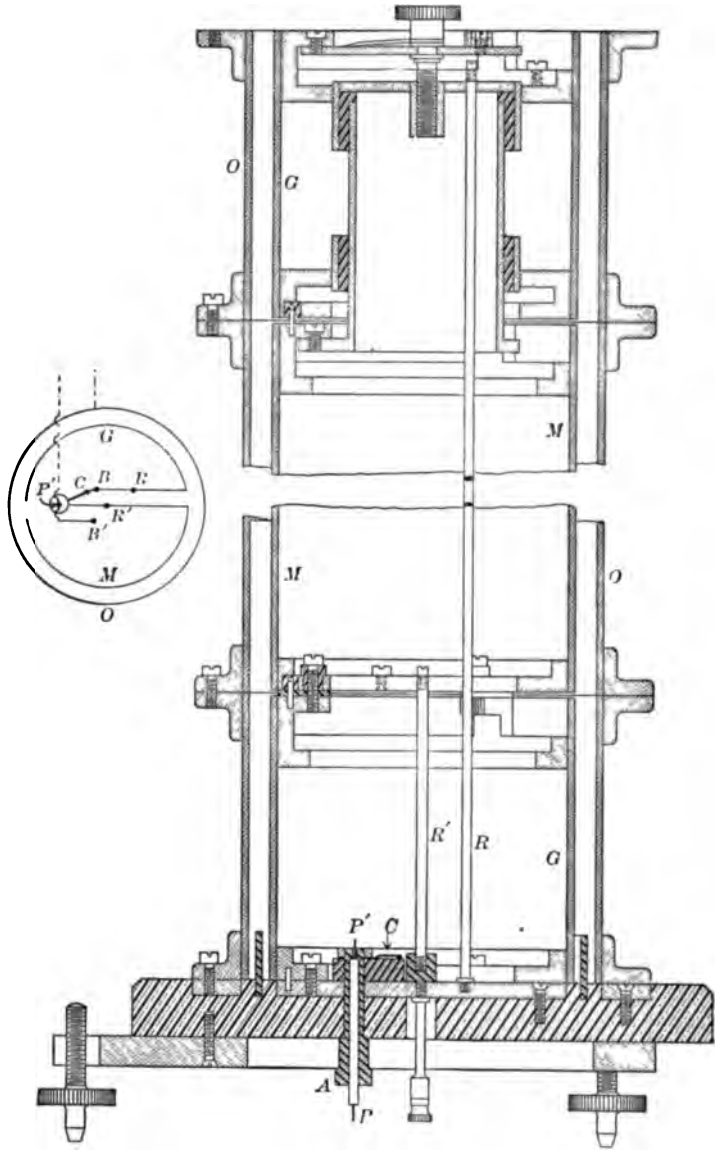

Fig. 5.—*Section of Guard Cylinder Condenser.*

The guard sections G are connected by the rod R, the upper section can be moved by the micrometer screw so as to regulate the width of the gap. The middle section M is connected to the rod R′, which in turn is connected to the spring arm C (small figure), which is rotated by the knob A so as to make contact with either B or B′, thus connecting the middle section either to the guard section or to the lead from the commutator.

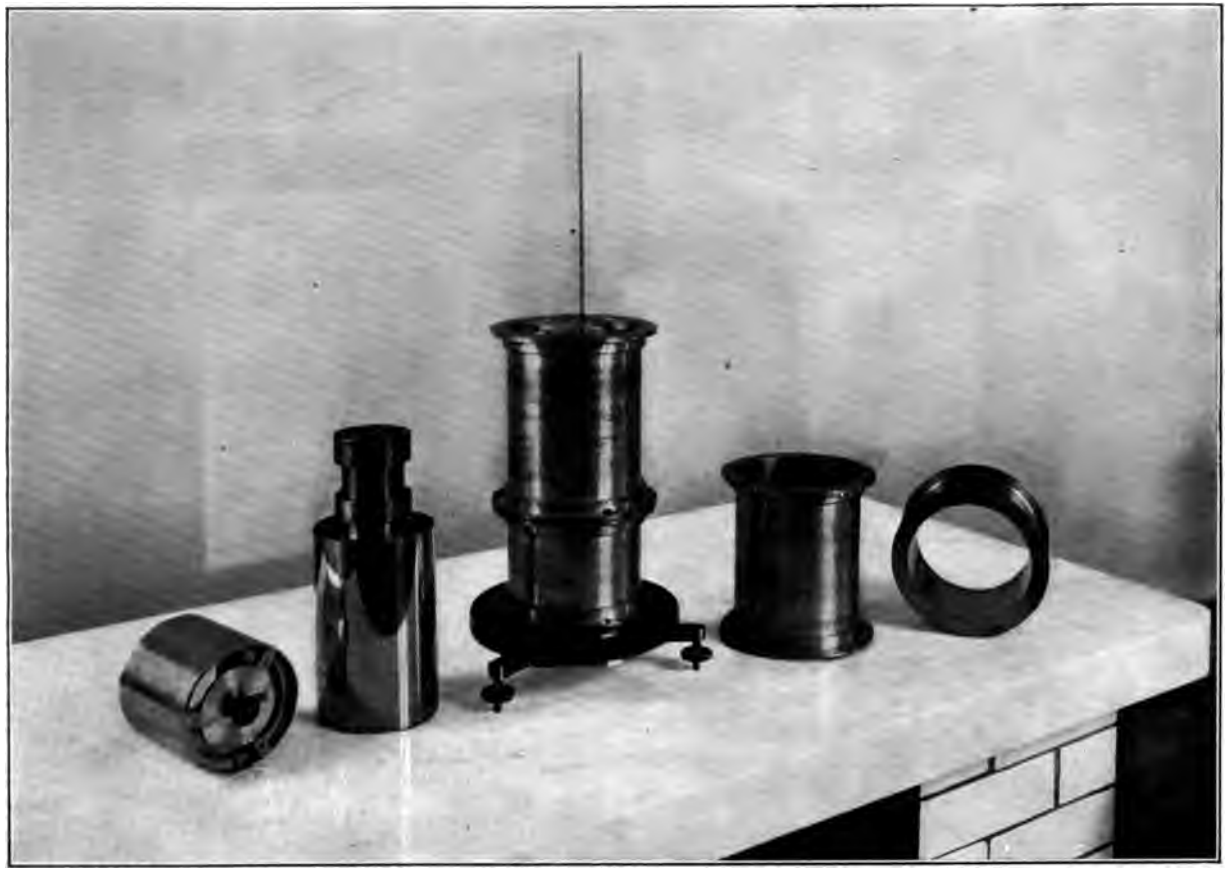

Fig. 4a.—*Cylindrical Condenser Partly Knocked Down.*

On the extreme left is shown the upper cylinder; next to it is seen inner cylinder No. 3 with the attachment which serves to guide the guard cylinder as it is raised or lowered. The corresponding outer cylinders are shown on the right. The rod extending above the lower sections serve to connect the upper and lower guard cylinders.

[3607–07. To face page 448.]

The cylinders were then carefully turned to size in a precision lathe, after which they were silvered and buffed down. Each inner cylinder was then placed coaxial with its corresponding outer one, and the ends of the combinations thus formed were ground until the two ends could be closed water-tight by means of flat ground glass plates.

These cylinders were intended to be built up to different heights so that the end effects could be eliminated by a method of differences, according to the plan since outlined by Lord Rayleigh.[3]

11. THE GUARD CYLINDERS.

When it was decided to use guard cylinders, a fourth section was added. These cylinders are of course of the same style as those just described, but the attachment of the inner cylinder is peculiar. To the top of the adjoining section is rigidly fastened a smaller brass tube, attached to which, but insulated from it, is a broad, carefully turned brass ring (Fig. 5). To the top of this tube, but insulated from it, is a second broad ring closed at the top with a heavy plate, in the center of which is a hole 5 mm in diameter threaded to fit a well-made screw of ½ mm pitch. Attached to the inside of the inner guard cylinder are a pair of rings, which pass with almost a piston fit over the broad rings mentioned. This enables the guard cylinder to be moved along its axis without lateral displacement. The top of the guard cylinder is closed by a brass plate having a divided circular scale; through the center of this plate and attached to it by a collar passes the screw of ½ mm pitch of which we have spoken. In this way the cylinder can be raised or lowered by the screw and its exact position can be determined from the position of a pointer attached to the screw and moving over the scale. This is for the purpose of studying the variation of the capacity with the width of the gap between the cylinder and its guard cylinder. The guard cylinder is prevented from turning with the screw, as it is raised and lowered, by means of a steadying pin attached to the adjoining cylinder and passing through an ebonite block. The lower guard cylinder was insulated from its neighbor by means of three ebonite plates 1.6 mm thick placed between the rings attached to the inside of the cylinders. The rings were turned down until when the blocks are in place the gap between the two sections of the cylinders is 0.6 mm. The screws and steadying pins for holding these sections together are insulated from the upper section by means of ebonite bushings. The ebonite plates do not come within 3 mm of the outer

face of the cylinders; the bushings for the screws and pins are still farther back.

For charging the guard-ring condenser a key of the form shown in Fig. 5 was employed.

12. SWITCH AND CHARGING WIRES.

The lead from the commutator is stretched taut and attached to an ebonite post, screwed to the table and vertically below the point P (Fig. 5). From this post it is stretched tightly to the point P, to which it is soldered, forming thus a prolongation of PP', so that when PP' is rotated by means of the ebonite head A the wire will be twisted about its axis, but will not be displaced with reference to other objects. P' is connected to the button B'. The spring arm C which rotates with PP', but is insulated from it, is connected with the rod R', which is connected with the middle section M of the inner cylinder. The other button, B, with which C can be brought into contact, is connected with the rod R, which connects the two guard cylinders G.

Hence when C rests upon B' the middle sections of the inner cylinder are connected through PP' with the lead to the commutator, and the capacity measured is that of the commutator, the lead, and the condenser. On the other hand, when C rests on B the button B' is insulated, and the middle sections of the inner cylinder are connected with the guard cylinders, and so are charged to the same potential as before, but by way of the guard-ring section of the commutator. Hence the capacity now measured is that of the commutator and leads only. The capacity of the latter must be the same as it was in the former case, for the only change that has been made is the rotation of C from B' to B, and these portions of the system are entirely inclosed by the inner cylinder, all of which is at the same potential in both cases. They are, therefore, always uncharged, and so contribute nothing to the capacity in either case. The leakage from B' to B and to C was frequently tested, and in all cases was found to be negligible.

13. PLATE CONDENSER.

This condenser was of the guard-ring type, and, as already stated, was built in the instrument shop of the Bureau. The details of its construction are shown in Fig. 7. The top of the instrument con-

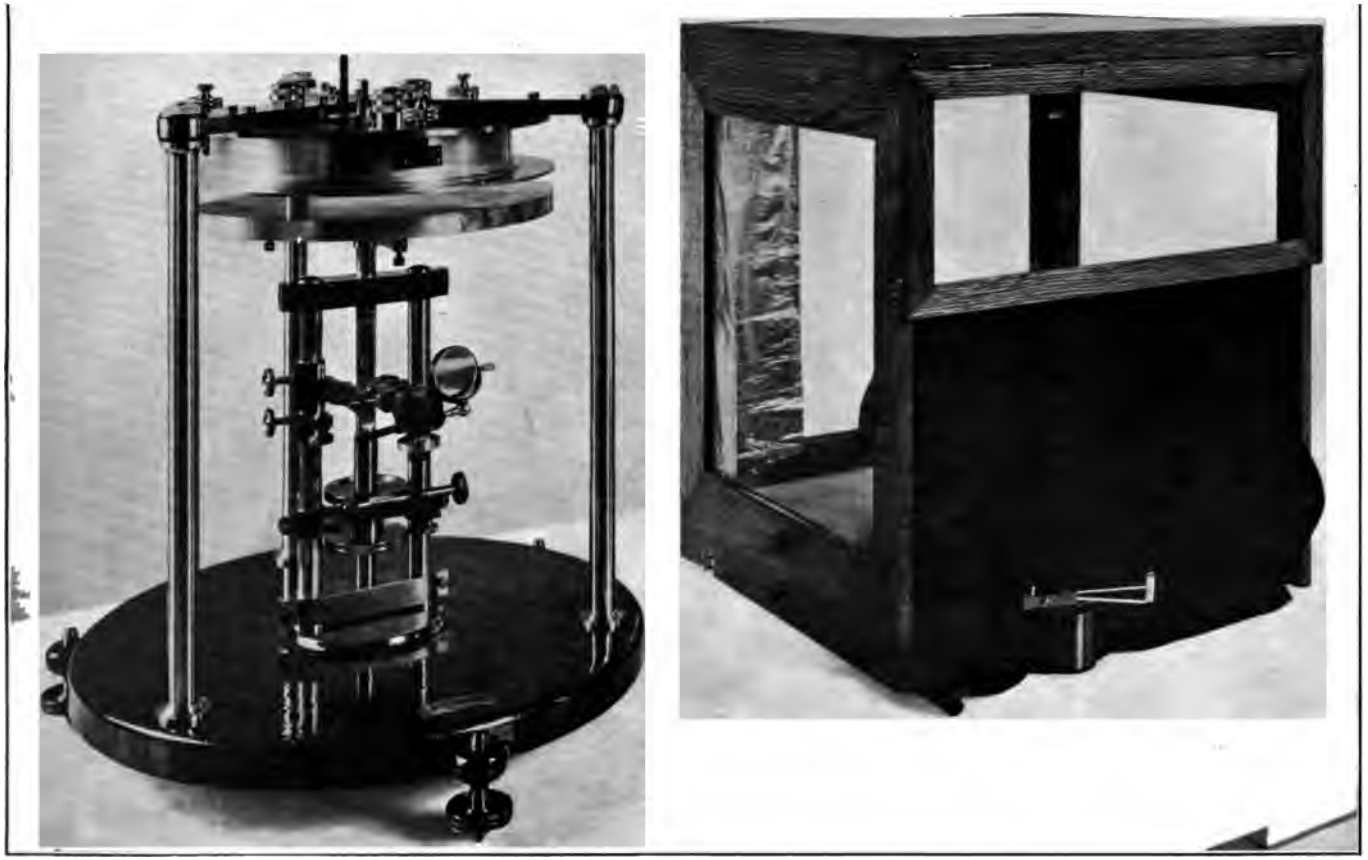

Fig. 6.—*Plate Condenser.*

To the right of the condenser is the case which fits over the condenser and within which the temperature can be maintained constant by electrical heating. Part of the front of the case is closed by a black felt curtain through which projects the eye piece of the reading microscope, and in which are two slits through which the hands can be passed for the purpose of moving the lower plate of the instrument. The back left-hand corner of the case is covered on the inside with tin foil, which is earthed so as to make the field around the leads (which pass up this corner of the case) as constant as possible. In front of the case may be seen the lever for adjusting the plates.

sists of a brass plate 6 mm thick and 30 cm in diameter; this is supported on three steel columns by means of heavy brass blocks 3.5 cm wide and 1.2 cm thick, screwed securely to the top of the plate and extending nearly to its center. The blocks are insulated

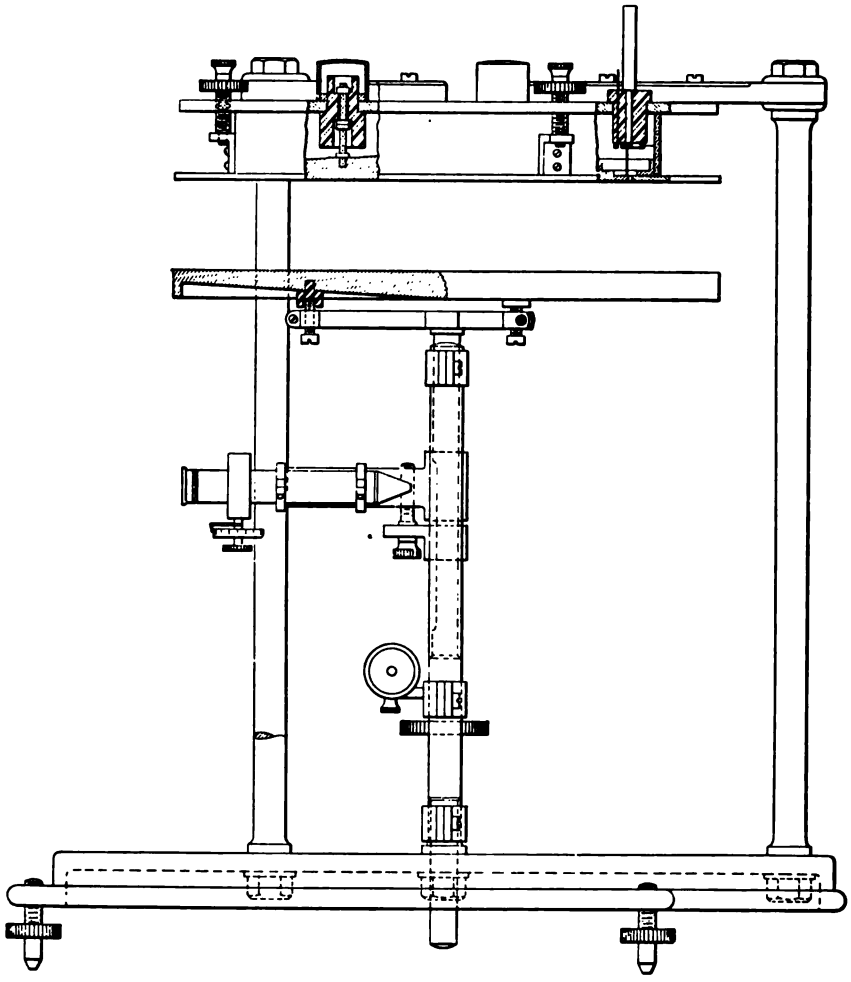

Fig. 7.—*Plate Condenser showing Manner in which the Guard Ring and Plates are Supported and Adjusted, Switch for Connecting the Collector Plate to either Guard Ring or Charging Wire, and the Microscope for Determining the Position of the Lower Plate.*

from the steel pillars by ebonite bushings. By means of three screws passing through ebonite bushings in this plate, the insulated plate of the condenser is supported below it. Lock nuts make the

position of this plate definite. Three other lock-nut screws passing through the top plate support the guard ring and enable us to adjust the latter to the plane of the insulated plate. Attached to the top of the guard ring is a short cylinder having a radius of about 3 cm less than that of the ring and of such a height as to reach nearly to the top plate of the instrument; it falls short of reaching the top plate by an amount just sufficient to allow the proper adjustment of the guard ring. This cylinder, with the top plate of the instrument, completely screens the top face of the insulated plate.

14. VARYING THE CAPACITY.

The other plate is supported under these on ebonite blocks on the three leveling screws of an inverted tripod. The tripod is screwed rigidly to a steel rod 1.5 cm in diameter. This rod has at its lower end a screw of 0.5 mm pitch and passes through guides so that it can be moved slowly and accurately along its own axis. To it is fastened a fine silver scale which can be read by means of a micrometer microscope rigidly attached to the base of the instrument.

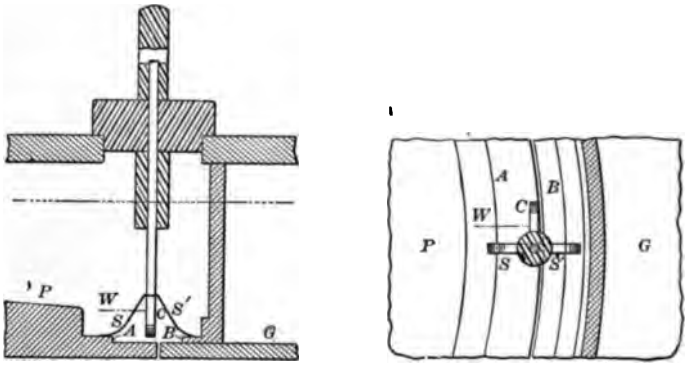

Fig. 8.—*Switch for Connecting Collector Plate P with either the Guard Ring G, or with the Wire W Connected with the Commutator.*

As shown the plate and ring are connected; by rotating the switch 90°, C makes contact with A and the plate is connected to the commutator.

By this means we can vary the distance between the plates by accurately known amounts. The entire instrument is placed inside a glass case in which the temperature can be controlled. The plates are of silvered brass and are ground plane; their backs are heavily ribbed so as to give them great rigidity. The instrument is provided with two collector plates of slightly different radii, so that two widths of guard ring gap can be employed.

15. SWITCH AND CHARGING WIRES.

The condenser is charged by means of a special key shown in Fig. 8. The wire from the commutator is stretched tightly to a stiff wire passing through an ebonite bushing set in the center of the top plate of the instrument. From the lower end of this wire, which is within the space entirely inclosed by the top plate of the condenser, the collector plate, and the guard ring, a spiral W of fine double-silk covered copper wire goes to the point C of the key. The key is situated immediately over the gap between the ring and the plate; it passes through an ebonite bushing in the top plate and is actuated by means of an ebonite handle. The guard ring and the collector plate are cut away near their edges, as shown in the figure, so that when the key is turned in the proper way and depressed the spring C makes contact with the plate, while the ends of the spring SS′ lie in the gap between A and B. In this position the plate and the guard ring are separately charged. If now the key be turned by 90°, the spring C will lie in the groove between A and B, while the other spring SS′ will touch A and B, thus connecting the plate and the guard ring; both will now be charged by means of the guard ring lead. The only portion of the lead to the collector plate which differs in the two cases is the portion of the key which is entirely surrounded by the collector plate, guard ring, and top plate of the condenser. Being in a region of uniform potential, this portion is uncharged. Hence the capacity of the main lead is the same in the two positions, and the difference in the capacities measured with the key in the two positions is exactly equal to the capacity of the collector plate of the condenser. Later a key similar to that employed for the cylinders was used.

16. ADJUSTING THE CONDENSER.

To adjust the condenser, the collector plate and guard ring are first adjusted, so as to lie approximately in a plane parallel to the top plate of the instrument and with the center of the collector plate coinciding with the center of the guard ring. When they are parallel to the top plate of the instrument they will be normal to the direction of motion of the rod carrying the lower plate. The centering of the plate is tested by measuring the width of the gap

between the plate and the ring at three points by means of a micrometer microscope, which can be placed in any one of three holes in the top plate and over the gap. The coincidence of the planes of the plate and of the ring was tested by a delicate lever, whose base rested on the lower plate of the condenser and could be slid so that the shorter arm of the lever touched either the plate or the ring. In this way the planes of the plate and of the ring could be brought into coincidence to within 0.005 mm., which suffices to reduce the error due to this cause to 3 parts in 10,000 in the most unfavorable case. Having adjusted the plate and the ring, the nuts on the screws were tightly locked and the upper plate with the collector plate and guard ring attached was removed from the steel pillars supporting it, turned upside down, and the width of the gap between the ring and the plate in the plane of the face of the condenser was measured at several equidistant points. The top plate with its attachments is then replaced and the lower plate of the condenser is raised to a suitable position, and the short arm of an accurate steel square is rested on it, and the distance between its supporting rod and the pendant long arm of the square is measured at two points as far apart as possible. This is repeated with the short arm of the square in a position at right angles to its former position. These measurements afford sufficient data to enable us to determine the angle the rod makes with the plane of the plate and to correct our results for it, if necessary. The coincidence of the planes of the ring and of the plate was occasionally tested.

When in use, the various holes in the top plate of the condenser, whether closed with ebonite bushings or not, are closed with metal caps having only the openings necessary to admit the leads.

No attempt is made to measure directly the true distance between the plates. Instead of this the plates are placed close together, say, 1 mm apart, and the capacity is measured. From this and an approximate value of ν the true electrical distance between the plates is calculated. The distance between the plates is now increased by a measured amount and the capacity is again measured. From this capacity and the total distance between the plates (the small electrical distance calculated plus the measured increase in the distance) a more approximate value of the ratio is calculated. If necessary, this better value of ν may be used to recalculate the

initial distance, and so a second approximation may be obtained. The capacity is actually measured for many different distances between the plates.

III. ELECTROSTATIC CAPACITIES.

17. SPHERICAL CONDENSER—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser bounded by two concentric spherical surfaces of radii R and r, taking the dielectric constant of the medium as unity, is

$$C = \frac{Rr}{R-r}$$

It is necessary, therefore, to determine accurately the mean interior radius of the hollow spherical shell, and the mean radii of the two spherical balls that are used in turn within the shell. The determination of these radii was made by two different methods. In the first, or gravimetric method, the volume of the shell is determined by ascertaining the mass of water it contains at a particular temperature, and the volume of the balls by finding the mass of water they displace when submerged. In the second method measurements of their diameters are made directly on the balls. On the shell accurate direct measurements can not be made. We have a check, however, on the gravimetric determination of the radius of the shell in the fact that if it were appreciably in error the results by the two balls would differ, since the error would affect the capacity when using the large ball more than when using the small one. The close agreement of the results by the two balls shows that there could be no serious error in the radius of the shell as determined by the gravimetric method.

a. **The Shell.**—The two hemispheres of the shell were screwed together in the position in which they were to be used in the determination of the electromagnetic capacity, and the joint, which was practically water-tight, was made perfect by means of a very little white lead around the joint on the outside of the shell (1904) or by means of a very little vaseline spread on the outermost edges of the flanges before the hemispheres were put together (1905). The shell thus prepared was weighed by the method of substitution. It was then filled with distilled water, care being taken to remove as completely as possible the last traces of air bubbles; this was done by

tipping the shell rather violently from side to side and by scraping the inner surface of the filled shell with a bent copper wire. After the shell had been filled level with the top of the hole in the upper hemisphere, it was weighed by the substitution method. A little of the water was then removed by means of a pipette, and the shell was refilled as before and again weighed. Finally the shell was emptied and dried and the process repeated. The temperature of the water was determined just before the completion of each filling, and again after each weighing. From these weighings the volume is computed and from this the radius is determined. The shell and weights both being of brass the correction to vacuo in the case of the empty shell is zero. A specimen determination of the volume of the shell is given below.

June 18, 1904.

Dry Bulb 22°.4 Barometer 76.36 cm; at 25°.9
Wet " 20°.6 −.79 " correction for temperature, humidity, etc.
75.57 " corrected pressure.

Shell filled, $t = 20°.77$, weighs 26 lbs. + 8596.631 g
" empty " 26 lbs. + 98.869 "
Apparent weight of water = 8497.762 "

Approximate volume of water = 8520 cc
" " of weights = 1010 "
Resultant volume air displaced = 7510 "

Log dens. air at 22°.4, 1 mm press. = $\bar{6}.1965$
Log 755.7 = 2.8783
" 7510 = 3.8756
Log weight w of air displaced = 0.9504 $\therefore w = 8.920$ g
$\therefore$ Mass of water = 8506.68 g

Log 8506.68 = 3.9297601
" D at 20°.77 = $\bar{1}.9991627$
" Vol. at 20°.77 = 3.9305974 $\therefore$ Volume = 8523.097 cc
" $\frac{4}{3}\pi$ = 0.6220886
" R^3 = 3.3085088
" R = 1.1028363 $\therefore$ $R = 12.67174$ cm at 20°.77
0.00018 cor. to 20°.0
$R_{20} = 12.67156$ cm

As a further check one determination of the volume of the interspace between the larger ball (ball *A*) and the shell was made by an analogous method. This gave a value very closely agreeing with the other results.

b. **The Balls.**—A similar method was adopted for determining the radius of the balls. From the difference in their apparent weights when in air and when suspended in water, their volumes are determined and from these their mean radii are obtained. Owing to the fact that the balls are hollow they float in water; hence a sinker has to be attached in order to submerge them. In 1904 the sinker was attached to a wire harness inclosing the ball, but in 1905 it was fastened to a small hook screwed into the bottom of the ball. The weight of the ball in air was determined by the method of double weighings, and the arms of the balance were found sufficiently equal to necessitate no correction for the weighings in water. The method of reduction is shown below.

(1) *Weight of Ball B in Water.*

April 11, 1905.

Air temperature 21°.1 C Barometer 74.30 cm
Hydrograph 50 per cent −.38 " correction
=9.3 mm press. 73.92 " corrected pressure.

(ball *B* + sinker, both in water at 21°.05) + wire	=	54.853 g
(0.041 cc more wire + sinker) in water + wire	=	651.979 "
(ball *B* −0.041 cc) in water at 21°.05	=	−597.126 "
Correction for 0.041 cc	=	− 0.041 "
Correction for air displaced by 597 g of brass	=	+ 0.083 "
True weight of ball *B* in water at 21°.05	=	−597.084 "

(2) *Weight of Ball B in Air.*

Air temperature 21°.0 Barometer = 74.10 cm
Hydrograph 50 per cent −.37 " correction
=9.2 mm press. 73.73 " corrected pressure

Approximate volume	= 2927 cc
Vol. Weights	= 275 "
Difference	= 2652 "

Weight of ball B in air	=	2321.144 g
Weight of 2652cc of air	=	3.089 "
Weight in vacuo of ball B	=	2324.233 "
" " water at 21°05	=	597.084 "
Loss of weight in water	=	−2921.317 "

Log 2921.317	= 3.4655787	
" D at 21°05	= $\bar{1}$.9991360	
" Vol. at 21°05	= 3.4664427	∴ Volume at 21°05 = 2927.134 cc
" $\frac{4}{3}\pi$	= 0.6220886	
" r^3	= 2.8443541	
" r	= 0.9481180	∴ r = 8.87397 cm at 21°05
		0.00017 correction to 20°0
		r = 8.87380 cm at 20°0

The weights used were compared with the standards of this Bureau and the necessary corrections have been applied. The thermometer used in 1904 had its corrections determined just before it was used, but owing to an oversight that used in 1905 was not standardized until 14 months after it was used. This may explain the slight difference between the two sets of results shown in the following summary; a rise of zero of 0°11 C between the time the thermometer was used and the time it was standardized will account for the difference.

18. SUMMARY OF DIMENSIONS.

Shell; radius at 20°		Ball A; radius at 20°		Ball B; radius at 20°
1904	1905	1904	1905	1905
12.67156 cm	12.67134 cm	10.11799 cm	10.11794 cm	8.87380 cm
12.67151	12.67118	10.11811	10.11787	8.87378
12.67169	12.67149	10.11814	10.11790	8.87381
12.67159[6]	12.67140	10.11808		8.87380
	12.67151			
	12.67143			
	12.67143			
	12.67140			

It will be noticed that the 1905 values are a little lower than the values of 1904, and by nearly the same relative amount for both ball A and the shell.

If we assume the thermometer used in 1905 changed 0°.11 between the time it was used and the time it was standardized the values for 1905 become:

Shell = 12.67158 cm; ball A = 10.11806 cm; ball B = 8.87391 cm.

From the above values of the radii we obtain the following values for the capacities:

Ball A and shell as determined in 1904		C = 50.2102 cm
" " " " " " 1905		C = 50.2087 "
	Mean	= 50.2094 "
" " " " " " "	(Assuming change in thermometer)	C = 50.2098 "
Ball B " " " " " "		C = 29.6091 "
" " " " " " "	(Assuming change in thermometer)	C = 29.6093 "
" as determined in 1905; shell as determined in 1904		C = 29.6080 "

The last is evidently an unfair combination, involving as it does a differential error in the thermometers; but even this value of the capacity differs from the others by only 4 parts in 100,000, which corresponds to an error of 2 parts in 100,000 in **v**. Hence, if we assume that the capacities of the condenser with the two balls are as follows:

With ball A, C = **50.2095** cm at 20° C
" " B, C = **29.6092** " " "

we shall probably not be in error by more than 2 in 100,000 in the electrostatic capacity, and this error would affect the value of **v** by only 1 in 100,000.

The possible effect of the two small holes in the shell (through which the charging wires pass, and through one of which the cord for suspending the ball passes) will be discussed below.

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[6] The following values of the radii were found in 1889 (E. B. R., loc. cit.):
Ball A = 10.1180 cm at 17°.0 C = 10.1185 cm at 20°.0 C
Ball B = 8.8735 " " 16°.5 C = 8.8741 " " "
Shell = 12.6805 " " 18°.4 C = 12.6811 " " "
These differ from those values just given by 5μ for ball A, 3μ for ball B, and 96μ for the shell. The differences in the case of the balls are less than the error allowed in the former work; the large difference in the case of the shell is due to the fact that before beginning the present work a few elevations on the flanges of the hemispheres were ground down so that the two hemispheres should make good contact all around the circumference, thus reducing the mean radius of the shell.

In the above reduction D is the specific gravity of water referred to water at 4°.0 C as unity as given by Chappuis.[7] If we assume that the density of water at 4°.0 C is 0.999955 as given by Guillaume,[8] then the volumes as given above are too small by 45 parts in 1,000,000, the electrostatic capacities are too small by 15 parts in 1,000,000, and the value found for v by using the assumed capacities will be too small by 7 parts in 1,000,000.

19. SPHERICAL CONDENSERS—DIRECT MEASUREMENTS.

In order to obtain an idea as to the figuring of the balls, and to get an independent check on the dimensions determined gravimetrically each ball was measured along 50 diameters. For this pur-

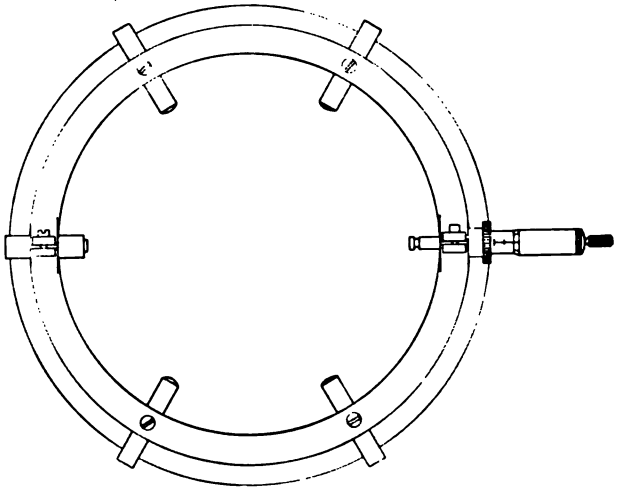

Fig. 9.—*Ring Caliper for Measuring Spheres.*

The four guides are adjusted so that the micrometer screw and its anvil lie along a diameter of the ball and the ball is just free to turn.

pose a ring caliper such as is shown in Fig. 9 was used. It was so constructed that when it and the ball to be measured were in position and resting upon a flat surface the micrometer screw and its

[7] Dilatation de l'Eau, Travaux et Memoires du Bureau International, vol. 13, pp. 1–40; 1904.

[8] Rapports Int. Cong. Paris, 1900, vol. 1, p. 99.

anvil lay in the equatorial plane, and the four guides lay along a circle of latitude of the ball. The guides were so adjusted that the ball could just turn freely, and so that the screw and its opposing anvil lay along a diameter. Diameters were measured every 18° around the equator, giving 20 measurements of 10 different diameters. Similar measurements were taken along four meridian circles differing in longitude by about 45°. Taken in this manner the diameters are not equally spaced over the surface of the sphere but are crowded at the poles and spread out at the equator, the spacing being inversely proportional to the sine of the polar angle. Hence in obtaining the mean diameter we must give each diameter a weight proportional to the sine of its polar angle. No measurement is taken nearer than 9° to the pole. In the tables below are given the differences "Diameter — End Standard" (in millimeters) for the various diameters measured; these have been corrected for a slight variation in temperature during the measurement.

The mean diameter of ball *A* at 24°.6 is equal to the length of the end standard plus the difference between it and the diameter of the ball, this difference being (as shown on the next page) 0.0609 mm the mean diameter of ball *A* is

$$= 20.23132 + 0.00609 = 20.23741 \text{ cm}$$

$$0.00167 \text{ " = correction to 20°.0}$$

$$\text{Mean diameter ball } A = 20.23574 \text{ " at 20°.0}$$

$$\text{" radius " } = 10.11787 \text{ " " "}$$

This differs from the value found gravimetrically by about 2μ, which is very fair agreement. For this ball the range is from 0.122 to $-0.031 = 0.153$ mm. This is a rather large quantity, but will not affect the final result appreciably.

TABLE I.

Measurements of Diameters of Ball **A** at 24°.6; Temperature of Standard 25° C.

Meridian position	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
0°	0.107	0.043	0.069	0.032	0.028	0.047
	0.111	0.109	0.102	0.073	0.051	0.068
	0.096	0.031	0.045	0.041	0.072	0.090
	0.110	0.115	0.114	0.115	0.098	0.100
						0.101
45°	0.087	0.103	0.076	0.070	0.074	0.086
	0.120	0.118	0.122	0.107	0.099	0.030
	0.049	0.100	0.071	0.072	0.100	0.001
	0.038	0.119	0.115	0.113	0.110	−0.028
						−0.020
90°	0.041	0.118	0.071	0.054	0.040	0.021
	0.032	0.103	0.080	0.078	0.062	0.067
	0.108	0.059	0.040	0.055	0.058	0.098
	0.020	0.108	0.078	0.079	0.070	0.101
						0.053
135°	−0.010	0.102	0.106	0.069	0.020	0.012
	0.027	0.060	0.022	−0.020	−0.019	−0.015
	0.012	0.101	0.067	0.049	−0.005	−0.031
	−0.025	0.064	0.029	−0.010	−0.024	0.021
						0.040
Mean δ	0.0577	0.0908	0.0754	0.0611	0.0521	0.0421
sin ϕ	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
δ sin ϕ	0.0090	0.0412	0.0533	0.0544	0.0515	0.0421

$\Sigma \sin\phi = 4.1962$; $\Sigma \delta \sin\phi = 0.2515$

$$\therefore \frac{\Sigma \delta \sin\phi}{\Sigma \sin\phi} = \frac{0.2515}{4.1962} = 0.0600 \text{ mm}$$

0.0009 " = cor. to bring standard

0.0609 " to 24°.6

TABLE II.

Measurements of diameters of Ball B, at 23°.4 C. The values given in the table are the differences δ between the diameters and the length of the end standard.

Meridian positions	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
0°	+0.010	+0.007	+0.012	+0.003	−0.020	−0.022
	−0.014	−0.034	−0.053	−0.090	−0.043	−0.035
	+0.002	+0.011	+0.014	+0.017	−0.010	−0.047
	+0.022	−0.009	−0.051	−0.074	−0.045	−0.053
						−0.054
45°	+0.013	+0.020	+0.018	−0.004	0.000	−0.055
	+0.009	+0.019	−0.020	−0.022	−0.017	−0.061
	+0.018	+0.018	−0.007	+0.001	−0.018	−0.073
	+0.010	+0.019	+0.008	−0.020	−0.027	−0.073
						−0.054
90°	+0.008	+0.013	+0.015	−0.006	−0.005	−0.031
	+0.013	−0.008	−0.016	−0.040	−0.026	−0.029
	+0.016	+0.010	+0.012	−0.007	−0.005	−0.052
	+0.018	+0.005	−0.013	−0.032	−0.026	−0.062
						−0.052
135°	+0.009	+0.003	+0.010	−0.009	−0.027	−0.072
	+0.014	−0.027	−0.039	−0.054	−0.025	−0.091
	+0.009	+0.008	+0.010	−0.017	−0.022	−0.057
	+0.010	+0.026	−0.022	−0.039	−0.054	−0.035
						−0.028
Mean δ	+0.0104	+0.0050	−0.0076	−0.0246	−0.0231	−0.0518
sin ϕ	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
δ sin ϕ	+0.0016	+0.0023	−0.0054	−0.0219	−0.0228	−0.0518

$$\Sigma \sin\phi = 4.1962;\ \Sigma\, \delta.\sin\phi = -0.0980$$

$$\therefore \frac{\Sigma\, \delta.\sin\phi}{\Sigma \sin\phi} = \frac{-0.0980}{4.1962} = -0.0234 \text{ mm.}$$

Hence mean diameter of ball B at 23°.4 C
$= 177.5$ mm end standard -0.0234 mm
$= 17.75116 - 0.00234 = 17.74882$ cm
0.00109 " correction to 20°.0 C
Mean diameter of ball $B = 17.74773$ " at 20°.0
$\therefore$ " radius " " " $= 8.87386$ " " "

This agrees with the value found by the gravimetric method to within 0.6μ, which is as close an agreement as can be expected. The extreme range of the settings is from -0.091 to $+0.026 = 0.117$ mm. This is not large enough to affect our results appreciably.

We regard the results by the gravimetric method as much more reliable than the direct measurements, and hence take the latter only as a check on the others.

It is not possible to study the figuring of the shell with the same thoroughness as we have the balls, but by bringing the ball into contact with the top of the inner surface of the shell and then lowering it until it touches the bottom of the shell we can measure the difference between the polar diameter of the ball and the vertical diameter of the shell. By this method we find—

Vertical diameter of shell	=	25.326 cm
Mean " " "	=	25.343 "
Difference	=	0.017 cm

We shall see later (page 476) that this error in figuring will not affect the final result.

20. CYLINDRICAL CONDENSERS—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser, consisting of two coaxial circular cylinders of radii R and r and length l, taking the dielectric constant of the medium as unity, is

$$C = \frac{l + \delta l}{2 \log_e \frac{R}{r}} \qquad (1)$$

where δl is the correction to the length due to the end effect. If

end cylinders are used as guard cylinders the value of δl can be calculated, and by varying the gap between the middle cylinder and the guard cylinders the calculated value of δl can be verified experimentally. This we have done. If there are no guard cylinders, δl can be eliminated by varying the length l, *provided care is taken to make the end conditions identical for the two cases*, so that δl will have the same value in each case. This is a very important matter, and involves many precautions.

In order to calculate the capacity C with high accuracy it is of course necessary, in addition to ascertaining or eliminating δl, to determine l, R and r with corresponding accuracy. Since

$$C = \frac{l}{2 \log_e \frac{R}{r}} = \frac{lr}{2(R-r)} \text{ approximately}$$

it is evidently necessary to get the difference of the radii $R-r$ with the same precision that is required for r. This makes it undesirable to have this difference very small. If it is 1 cm, this difference must be known to 0.5 micron to give C to within 1 in 20,000. As in the case of the spherical condenser, the radii were determined by two independent methods, (1) by gravimetric observations and (2) by direct measurements.

It ought to be stated that at the time these cylindrical condensers were built the Bureau instrument shop possessed no grinding machine, and the cylinders were not as accurately turned as could be wished. We had not at that time set so narrow a limit of tolerance as we came to later, and hence what was satisfactory then was not entirely satisfactory a year afterwards. We have, however, taken so large a number of measurements on the cylinders that we believe the mean values obtained are thoroughly trustworthy.

The cylinders were ground on their ends in pairs, so that when placed one within the other the two ends of the space between the cylinders could be closed water-tight by a pair of slightly greased, flat, ground-glass plates. By a procedure exactly analogous to that employed in the case of the shell the volume of the outer cylinder and the volume of the interspace between the two cylinders were determined. In every case the cylinders were closed at the top with a flat, ground-glass plate fitting water-tight and containing two holes

to facilitate the filling. The filling was always such that the meniscus rising from these holes was estimated to be just sufficient to fill the holes level full. During the weighing metal caps were placed over the holes in order to reduce the evaporation. As the space to be filled was closed with glass at both top and bottom, the entrapped air bubbles could be readily seen and (with patience) removed.

By means of an end standard comparator the length of each cylinder along four generating lines was measured by the division of weights and measures of this Bureau. Though the ends of the cylinders were ground in pairs with the cylinders approximately coaxial, the inner cylinder of each pair was found to be about 7μ longer than the corresponding outer one. This is not sufficient to cause trouble from leakage, but needs to be considered in the determination of the radii. Since the capacity of concentric cylinders depends upon the *ratio* of their radii, it will be unaffected by any error in the absolute density of water, or in the thermometer, provided these errors are the same in the determination of the volume of the interspace between the cylinders as in the determination of the volume of the outer cylinder. The dimensions found were as follows:

Volume, outer cylinder No. 2 at 20°.0 C
= 3297.033 cc; length = 19.99875 cm

Volume, interspace, cylinders No. 2 at 20°.0 C

= 836.897 cc, weight 2
= 836.822 " " 1
836.872 " weighted mean

; length = 19.99946 cm

Volume, outer cylinder No. 3 at 20°.0 C

= 3293.167 cc
= 3293.114 "
3293.140 "

; length = 20.00718 cm

Volume interspace, cylinders No. 3 at 20°.0 C

= 832.110 cc
= 832.083 "
832.096 "

; length = 20.00768 cm.

From these we find:

Mean radius outer cylinder No. 2 at 20°.0 C = 7.24411 cm
" " inner " " " " " = 6.25760 "
" " outer " No. 3 " " = 7.23831 "
" " inner " " " " = 6.25740 "
Capacity per unit of length of No. 2 = 3.41549 cm
" " " " " " No. 3 = 3.43350 cm.

The charge measured being that on the *inner* cylinder, the effective capacities of the cylinders are the capacities per unit length multiplied by the lengths of the *inner* cylinders. Hence we have for the capacities from formula (1), not including any corrections for guard-ring gap or end or edge effects, the following values:

Capacity of cylinders No. 2 = **68.3080** cm at 20°.0
" " " No. 3 = **68.6965** " " "

21. CYLINDRICAL CONDENSERS—DIRECT MEASUREMENTS.

As in the case of the spheres, the direct measurements were taken partly to obtain a check on the more accurate gravimetric determinations and partly to determine the irregularities in the cylinders. The diameters of the outer cylinders were measured at angular intervals of 22°.5 at eight or ten sections of the cylinders. The results are as follows:

TABLE III.

Measurements of Diameters of Outer Cylinder No. 2.

Angular position	Difference; Diameter—Standard. 1 unit = 0.4617 μ									Temperature	Correction to 20° C	Difference at 20° C
	0°	22°.5	45°	67°.5	90°	112°.5	135°	157°.5	Mean			
1. At top	82.7	85.8	104.0	103.8	100.4	90.4	81.1	81.8	91.2	25°.6	−5.7 μ	36.4 μ
2. 2.5 cm from top	222.9	232.4	222.6	232.1	235.4	227.9	213.0	206.4	224.1	27°.8	−7.9 μ	95.6 μ
3. 5.0 " " "	218.4	223.2	223.2	224.8	230.2	219.0	214.4	213.3	220.8	25°.75	−5.8 μ	96.2 μ
4. 7.5 " " "	89.3	111.3	97.8	93.5	84.0	83.2	75.0	76.0	88.8	27°.8	−7.9 μ	33.1 μ
5. 10.0 " " "	60.8	68.6	68.8	55.6	49.8	44.8	41.8	49.2	54.9	26°.2	−6.3 μ	19.0 μ
6. 10 cm from bottom	62.0	68.2	61.6	48.0	45.0	43.5	44.0	49.2	52.7	27°.25	−7.4 μ	16.9 μ
7. 7.5 cm from bottom	51.6	54.5	41.9	24.4	14.7	19.6	32.8	34.0	34.2	27°.7	−7.8 μ	8.0 μ
8. 5.0 cm from bottom	66.1	69.6	51.2	41.6	27.3	16.6	21.6	47.8	42.7	28°.4	−8.5 μ	11.2 μ
9. 2.5 cm from bottom	157.2	168.6	146.6	120.2	103.2	106.0	113.6	149.8	133.2	27°.6	−7.7 μ	53.8 μ
10. At bottom	267.8	258.8	223.8	204.4	186.2	191.1	208.1	227.9	221.0	29°.25	−9.4 μ	92.6 μ

Weighting the top, bottom, and two middle values half as heavily as the others we find for the mean difference 47.5μ

Standard at 20°.0	= 14.48324 cm
Diameter—Standard	= 0.00475 "
Mean diameter No. 2	= 14.48799 "
" radius "	= 7.24400 "

This differs from the value found gravimetrically by 1.1μ. Considering that the diameter varies through a range of 117μ, this agreement is all that could be expected.

TABLE IV.

Measurements of Diameters of Outer Cylinder No. 3.

Angular position	Difference; Diameter—Standard. 1 unit = 0.4617μ									Temperature	Correction to 20°.0 C	Difference at 20°.0 C
	0°	22°.5	45°	67°.5	90°	112°.5	135°	157°.5	Mean			
1. At top	+103.0	+111.2	+121.4	+123.4	+125.8	+115.8	+108.3	+107.2	+114.5	27°.85	−8.0μ	+ 44.9μ
2. 2.5 cm from top	− 39.0	− 35.2	− 27.1	− 12.8	− 9.0	− 6.6	− 17.5	− 23.4	− 21.4	24°.90	−5.0μ	− 14.9μ
3. 5.0 cm from top	−131.8	−120.0	−106.0	−100.6	−105.9	−117.5	−136.2	−144.2	−120.3	28°.3	−8.4μ	− 63.9μ
4. 10 cm from top	−177.6	−201.6	−193.4	−177.9	−161.0	−152.8	−150.8	−162.0	−172.1	29°.2	−9.4μ	− 88.9μ
5. 10 cm from bottom	−165.0	−180.4	−183.0	−158.0	−149.3	− 133.9	−133.5	−145.1	−156.0	25°.8	−5.9μ	− 77.9μ
6. 5.0 cm from bottom	−224.9	−246.3	−242.8	−233.8	−225.0	−211.6	−196.5	−206.4	−223.4	26°.2	−6.3μ	−109.4μ
7. 2.5 cm from bottom	−219.6	−228.7	−241.0	−245.5	−221.8	−208.4	−196.6	−204.1	−220.7	25°.1	−5.2μ	−107.1μ
8. At bottom	− 10.8	− 8.6	− 4.8	− 5.9	− 3.8	+ 6.4	+ 6.8	+ 4.8	− 2.0	26°.4	−6.5μ	− 7.4μ

Weighting the end sections 1, the middle sections 3 each, the others 2, we find for the mean difference − 65.8μ.

Standard at 20°.0	= 14.48324 cm
Diameter—Standard	= −0.00658 "
Mean diameter No. 3	= 14.47666 cm at 20°.0 C
" radius "	= 7.23833 " " "

This differs from the value found gravimetrically by only 0.2μ.

The diameter of this cylinder varies over a range of 172μ.

Having found that the sections of the inner cylinders are very nearly circular, three diameters of each of 20 sections were measured. The screw micrometer employed was graduated to 0.01 mm.

TABLE V.

Measurements of Diameters of Inner Cylinder No. 2.

	Series 1.—Micrometer reading in mm				Series 2.—Micrometer reading in mm				
Diameter at—	0°	60°	120°	Mean	0°	60°	120°	Mean	Weight
Distance from top—									
0.5 cm	0.154	0.162	0.159	0.1583	0.158	0.155	0.154	0.1557	1
1.5 "	.139	.155	.144	.1460					
2.5 "	.140	.151	.140	.1437	.148	.140	.148	.1453	3
3.5 "	.140	.152	.141	.1443					
4.5 "	.141	.150	.141	.1440					
5.5 "	.139	.150	.139	.1427	.145	.138	.143	.1420	3
6.5 "	.140	.148	.138	.1420					
7.5 "	.141	.146	.134	.1403					
8.5 "	.144	.149	.140	.1443	.145	.139	.144	.1427	3
9.5 "	.148	.150	.149	.1490					
10.5 "	.142	.145	.146	.1443					
11.5 "	.144	.144	.144	.1440	.144	.132	.139	.1387	3
12.5 "	.140	.143	.140	.1410					
13.5 "	.139	.143	.140	.1407					
14.5 "	.138	.142	.140	.1400	.144	.135	.139	.1393	3
15.5 "	.140	.144	.147	.1437					
16.5 "	.140	.140	.139	.1397					
17.5 "	.130	.137	.137	.1347	.138	.130	.130	.1327	3
18.5 "	.137	.140	.140	.1390					
19.5 "	.169	.166	.168	.1677	.165	.159	.164	.1627	1
Mean				0.14447				0.14202	

RESULTS, SERIES 1.

Mean reading on diameter	+0.1445 mm
" " " end standard	−0.0061 "
" diameter—standard	+0.1506 "
Correction from 23°.7 to 20°.0 C	−0.0032 "
(Diameter—Standard) at 20°.0	+0.1474 "

SERIES 2.

Mean reading on diameter	+0.1420 mm
" " " standard	−0.0084 "
" diameter—standard	+0.1504 "
Correction from 24°.2 to 20°.0 C	−0.0037 "
(Diameter—Standard) at 20°.0	+0.1467 "

A third set gave (Diameter − Standard) at 20°.0 = +0.1503 mm.

The mean of all three is (Diameter − Standard) at 20°.0 = 0.1481 mm.

End-Standard = 124.9962 mm at 20°.0
Diameter—Standard = 0.1481 " " "
Mean Diameter No. 2 = 125.1443 " " "
" Radius No. 2 = 6.25722 cm " "

This is 3.8 μ smaller than that found by the gravimetric method.

TABLE VI.

Measurements of Diameters of Inner Cylinder No. 3.

	Series 1 Temp. 24°.0 Micrometer reading in mm				Series 2 Temp. 24°.6 Micrometer reading in mm			
Diameter at—	0°	60°	120°	Mean	0°	60°	120°	Mean
Distance from top:								
0.5 cm	0.156	0.156	0.152	0.1547	0.159	0.160	0.155	0.1580
1.5 "	.138	.147	.133	.1393	.139	.138	.141	.1393
2.5 "	.133	.144	.130	.1357	.135	.137	.140	.1373
3.5 "	.139	.146	.138	.1410	.140	.141	.140	.1403
4.5 "	.137	.149	.141	.1423	.143	.145	.148	.1453
5.5 "	.134	.149	.140	.1410	.141	.146	.147	.1447
6.5 "	.137	.145	.140	.1407	.143	.143	.146	.1440
7.5 "	.138	.145	.140	.1410	.141	.141	.146	.1427
8.5 "	.136	.146	.141	.1410	.142	.141	.148	.1437
9.5 "	.139	.150	.144	.1443	.147	.145	.150	.1473
10.5 "	.141	.157	.149	.1490	.149	.149	.155	.1510
11.5 "	.145	.155	.151	.1503	.151	.150	.159	.1533
12.5 "	.140	.155	.150	.1483	.149	.149	.159	.1523
13.5 "	.140	.151	.149	.1467	.148	.146	.158	.1507
14.5 "	.139	.152	.147	.1460	.148	.142	.154	.1480
15.5 "	.140	.150	.144	.1447	.145	.144	.150	.1463
16.5 "	.139	.150	.148	.1457	.148	.148	.153	.1497
17.5 "	.140	.150	.147	.1457	.142	.140	.151	.1443
18.5 "	.140	.151	.148	.1463	.141	.143	.150	.1447
19.5 "	.150	.161	.157	.1560	.150	.150	.159	.1530
			Mean	0.1450			Mean	0.1468
			Correction to 20°	−.0035			Correction to 20°	.0040
			Mean reading at 20°	0.1415			Mean reading at 20°	0.1428
			Mean of both series, 0.1422 mm.					

Micrometer reading on end-standard at 20°.0 = −0.0077 mm
" " " cylinder " " = +0.1422 "
Diameter—Standard = +0.1499 "
" = 124.9962 "
Diameter of inner cylinder No. 3 = 125.1461 "
Radius " = 6.25730 cm

This is 1.0 μ smaller than the value found by the gravimetric method.

In the same manner several diameters at the upper end of the lowest section (No. 1) and at the lower end of the upper guard section (No. 4) were measured. The results of these measurements, as well as of those at the extremities of the other sections, are given in the following table:

TABLE VII.

Mean Radius at Ends of Sections.

Section	1	2	3	4
Radius at top, outer cylinder	7.24192	7.24344	7.24386	
" " bottom, " "		7.24625	7.24125	7.23774
Radius at top, inner cylinder	6.25758	6.25785	6.25782	
" " bottom, " "		6.25832	6.25774	6.25675

From these measurements it is evident that if the ends of the inner cylinders were truly circular and if the sections were always placed truly coaxially and with top end uppermost the offset at the junction of any two sections can in no case exceed 11.0 μ. Under the same conditions the offset in the case of the outer cylinders may amount to 61.2 μ. The effect of these offsets in the combinations actually used will be discussed below in Section V.

22. PLATE CONDENSER—DIMENSIONS OF THE PLATES.

The electrostatic capacity of a guard-ring condenser consisting of parallel circular plates is

$$C=\frac{A}{4\pi d}=\frac{(r+\delta r)^2}{4d} \qquad (2)$$

if the dielectric constant of the medium between the plates is unity. Here r is the radius of the collector plate; δr is a term due to the effect of the gap between the collector plate and the guard ring, depending upon the width of this gap and the values of r and of d; d is the distance between the plates. As already described, d is varied over wide limits and is determined at the time the condenser is used. On the other hand, the value of r is a constant and can be measured once for all. Similarly the width of the gap between the collector plate and the guard ring, depending only upon the radius

of the plate and the internal radius of the guard ring, is a constant of the instrument.

The diameters of the two plates were measured with an end standard comparator. Nine equally spaced diameters were measured for each plate; the diameters were measured in both direct and reversed directions and two settings were made in each position. The results obtained are given in Table VIII.

TABLE VIII.

Diameters of the Plates at 20°.0 C.

Diameter	Plate A	Plate B
1	20.0080 cm	20.0354 cm
2	20.0086 "	20.0359 "
3	20.0094 "	20.0363 "
4	20.0094 "	20.0363 "
5	20.0092 "	20.0357 "
6	20.0086 "	20.0350 "
7	20.0082 "	20.0346 "
8	20.0076 "	20.0345 "
9	20.0076 "	20.0347 "
Mean	20.00851 "	20.03538 "

From these measurements we see that the plates are practically ellipses of semi-axes 10.00470, 10.00380 for plate A and 10.01824, 10.01725 for plate B. For plate A, $2\sqrt{ab} = 20.00850$ cm, for plate B, $2\sqrt{ab} = 20.03538$ cm. Hence the areas of the plates are practically the same as the areas of the circles with diameters equal to the mean diameters of the plates, so we may take these mean diameters as the values of r in the expression for the capacity of the condenser.

Plate B, having been adjusted as already described, the entire top of the condenser was removed, and turned upside down, and the width of the gap between the collector plate and the guard ring (in the plane of the face of the plate) was measured at six points around the circumference of the plate. The measurements were made by means of a micrometer microscope, one division of

the scale being equal to 1.708 microns. Six measurements were made at each point. The values found are given in Table IX.

TABLE IX.

Width of Guard Ring Gap, Plate B. One Unit=1.708μ, 20°.2 C.

Position	Width of gap						Mean
1	121.0	120.5	122.3	118.5	121.3	120.6	120.7
2	103.4	101.2	105.0	99.2	102.9	101.4	102.2
3	100.5	102.5	104.0	105.0	102.1	102.0	102.7
4	109.1	105.2	108.0	108.8	105.5	106.7	107.2
5	88.9	94.3	90.8	97.1	89.0	92.3	92.4
6	102.9	(110.2)	102.1	107.0	101.8	102.1	103.2
1	124.7	124.0	119.1	119.1	117.4	120.2	120.9

Hence the mean width of the gap is 104.7 divisions = 178.8 microns. It is evident that the opening in the guard ring is considerably more elliptical than the plates. This width of gap corresponds to a mean radius of 10.03557 cm for the inner boundary of the guard ring.

IV. CORRECTIONS AND SOURCES OF ERROR—SPHERICAL CONDENSERS.

23. EFFECT OF SHELL NOT BEING PERFECT SPHERE.

The capacity of the spherical condenser is calculated on the hypothesis that both ball and shell are truly spherical and accurately concentric. As these conditions are never exactly fulfilled, it is necessary to consider the possible error thus introduced. We shall begin with a study of the shell. We have determined the radius, R, of a sphere having the same volume as the shell, and by determining the distance through which we raise the ball in order to bring it from contact with the bottom of the shell to contact with the top of it we have measured the difference between the vertical diameters of the balls and of the shell. This was done for both balls, and the mean of these measurements showed that the vertical diameter of the shell is less than its mean diameter R by

0.017 cm. If we assume this error is due to the fact that the flanges where the hemispheres come together have been ground down too much, so that each half is somewhat less than a complete hemisphere, we shall not be far from the truth and shall obtain a correction that is probably greater than the true one. For a perfect spherical condenser the capacity is $\frac{Rr}{R-r} = \frac{Rr}{d}$ where R and r are the radii of shell and ball respectively and d is the difference in their radii. Now any small change in R or r affects Rr very slightly, but produces a relatively large change in d; hence in the study of the effect of *small* errors, we may regard Rr as known and limit ourselves to an investigation of the factor $1/d$. We also know that

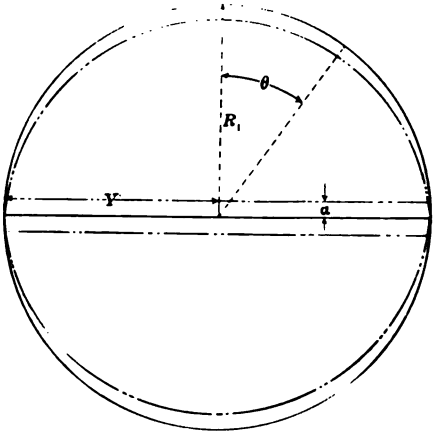

Fig. 10.

if the irregularities are slight the lines of force can depart but slightly from right lines normal to the surface. Hence from analogy with the parallel plate condenser we must give each element of $1/d$ a weight proportional to the area of the surface to which it applies. Now in the case in which d is the same for all points on a circular cone having its vertex at the center of the sphere, we may take $2\pi R^2 \sin\theta\, d\theta$ for our element of area so that for the d corresponding to the angle θ, the weight of $1/d$ will be

$$\frac{2\pi R^2 \sin\theta\, d\theta}{4\pi R^2} = \tfrac{1}{2} \sin\theta\, d\theta$$

Hence the total capacity will be

$$C = Rr \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{d_1}\, d\theta \qquad (3)$$

If d_1 is independent of θ this becomes $C = \frac{Rr}{d}$ as before.

Let R_1 = true radius of curvature of the shell, assumed constant,

R = radius of a sphere *having the same volume*,

a = the amount each hemisphere has been ground off on its flange beyond the true equitorial plane,

ϵ = the amount by which the vertical radius $R_1 - a$ falls short of the mean radius R; that is,

$\epsilon = R - (R_1 - a) = R - R_1 + a$

Write $\delta = R_1 - R$

Then $\epsilon = a - \delta$ or $a = \delta + \epsilon$

But the volume of the shell is

$$V = 2\pi \int_a^{R_1} (R_1^2 - x^2)\, dx = \frac{4}{3}\pi R_1^3 - 2\pi\left(R_1^2 a - \frac{a^3}{3}\right) = \frac{4}{3}\pi R^3$$

Substituting $R_1 = R + \delta$ we find, neglecting squares, products and higher powers of ϵ and δ

$$\frac{4}{3}\pi(R^3 + 3R^2\delta) - 2\pi(\delta + \epsilon)R^2 = \frac{4}{3}\pi R^3$$

$$\therefore 4R^2\delta = 2R^2(\delta + \epsilon)$$

$$\therefore \delta = \epsilon$$

$$\text{or, } a = \delta + \epsilon = 2\epsilon$$

The distance Y along the plane of union of the two hemispheres from the center of the ball to the shell is

$$Y = \sqrt{R_1^2 - a^2} = \sqrt{R^2 + 2R\delta + \delta^2 - a^2} = \sqrt{R^2 + 2R\epsilon - 3\epsilon^2}$$

$$= R\left\{1 + \frac{2\epsilon}{R} - \frac{3\epsilon^2}{R^2}\right\}^{\frac{1}{2}} = R\left\{1 + \frac{\epsilon}{R} \cdots\right\}$$

$\therefore Y = R + \epsilon$ approximately.

That is, the horizontal radius is greater than the mean radius R by ϵ, while the vertical radius is less than the mean by the same

amount. Hence, if d is the mean distance between the shell and the ball $(R-r)$ the distance between the shell and the ball along a radius of the latter making an angle θ with the vertical is:

$$d_1 = d + \epsilon - a\cos\theta$$
$$\text{or, } d_1 = d + \epsilon(1 - 2\cos\theta)$$

Substituting this value of d_1 in equation (3), we have for the true capacity of the ball

$$C = Kr\int_{\frac{\pi}{2}}^{\pi} \frac{\sin\theta\, d\theta}{d + \epsilon(1 - 2\cos\theta)} = Kr\int_{0}^{1} \frac{dz}{d + \epsilon - 2\epsilon z}, \text{ where } z = \cos\theta$$

$$= Kr\left[\frac{1}{2\epsilon}\log(d + \epsilon - 2\epsilon z)\right]_0^1 = \frac{Kr}{2\epsilon}\log\frac{d+\epsilon}{d-\epsilon}$$

$$\therefore C = \frac{Kr}{2\epsilon}\left[\begin{array}{l} \frac{\epsilon}{d} - \frac{1}{2}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^3 - \frac{1}{4}\left(\frac{\epsilon}{d}\right)^4 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^5 - \ldots. \\ + \frac{\epsilon}{d} + \frac{1}{2}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^3 + \frac{1}{4}\left(\frac{\epsilon}{d}\right)^4 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^5 + \ldots. \end{array}\right]$$

$$\therefore C = \frac{Kr}{d}\left[1 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^4 + \ldots.\right] \qquad (4)$$

But $\frac{Kr}{d} = C_0$ = the capacity calculated on the assumption that the figuring is perfect and that the dimensions are those of perfect spheres of the same volume. Writing $\frac{\Delta_1 C}{C_0} = \frac{C - C_0}{C_0}$ = relative increase in capacity produced by the irregularity, we have

$$\frac{\Delta_1 C}{C_0} = \frac{1}{3}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^4 + \ldots$$

But
$2\epsilon = 0.017$ cm
$d = 2.554$ cm for ball A
$d = 3.798$ " " " B

$$\therefore \frac{\Delta_1 C}{C_0} = 0.00000037 = 3.7 \times 10^{-6} \text{ for ball A}$$
$$= 0.00000017 = 1.7 \times 10^{-6} \text{ " " B.}$$

This increase in the capacity due to the shell not being perfectly spherical is entirely negligible. It has been computed on the

assumption that the center of the ball coincides with the center of figure of the shell. As this adjustment is never exact we shall proceed to investigate the effect of a slight displacement of the center of the ball from the center of figure of the shell.

24. EFFECT OF DISPLACEMENT OF THE BALL FROM THE CENTER.

Suppose the ball is raised by the amount β cm. Then for the upper hemisphere $d_1 = d+\epsilon-(2\epsilon+\beta)\cos\theta$ and for the lower hemisphere it is $d_2 = d+\epsilon-(2\epsilon-\beta)\cos\theta$.

The capacity then is

$$C=\frac{Rr}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{d+\epsilon-(2\epsilon+\beta)\cos\theta}+\int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{d+\epsilon-(2\epsilon-\beta)\cos\theta}\right]$$

Putting $\cos\theta = z$, we have

$$C=\frac{Rr}{2}\left[\int_0^1 \frac{dz}{d+\epsilon-(2\epsilon+\beta)z}+\int_0^1 \frac{dz}{d+\epsilon-(2\epsilon-\beta)z}\right]$$

$$=\frac{Rr}{2}\left[-\frac{1}{2\epsilon+\beta}\log\{d+\epsilon-(2\epsilon+\beta)z\}-\frac{1}{2\epsilon-\beta}\log\{d+\epsilon-(2\epsilon-\beta)z\}\right]_0^1$$

$$=\frac{Rr}{2}\left[-\frac{1}{2\epsilon-\beta}\log\frac{d-\epsilon+\beta}{d+\epsilon}-\frac{1}{2\epsilon+\beta}\log\frac{d-\epsilon-\beta}{d+\epsilon}\right]$$

$$=\frac{Rr}{2}\left[\begin{array}{l}\frac{1}{2\epsilon-\beta}\left[\begin{array}{l}\frac{\epsilon-\beta}{d}+\frac{1}{2}\left(\frac{\epsilon-\beta}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon-\beta}{d}\right)^3+\frac{1}{4}\left(\frac{\epsilon-\beta}{d}\right)^4+\cdots\\ +\frac{\epsilon}{d}-\frac{1}{2}\left(\frac{\epsilon}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon}{d}\right)^3-\frac{1}{4}\left(\frac{\epsilon}{d}\right)^4+\cdots\end{array}\right]\\ +\frac{1}{2\epsilon+\beta}\left[\begin{array}{l}\frac{\epsilon+\beta}{d}+\frac{1}{2}\left(\frac{\epsilon+\beta}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon+\beta}{d}\right)^3+\frac{1}{4}\left(\frac{\epsilon+\beta}{d}\right)^4+\cdots\\ +\frac{\epsilon}{d}-\frac{1}{2}\left(\frac{\epsilon}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon}{d}\right)^3-\frac{1}{4}\left(\frac{\epsilon}{d}\right)^4+\cdots\end{array}\right]\end{array}\right]$$

$$=\frac{Rr}{2}\left[\begin{array}{l}\frac{1}{2\epsilon-\beta}\left\{\frac{2\epsilon-\beta}{d}-\frac{1}{2}\frac{\beta(2\epsilon-\beta)}{d^2}+\frac{1}{3}\frac{\epsilon^3+(\epsilon-\beta)^3}{d^3}-\cdots\right\}\\ +\frac{1}{2\epsilon+\beta}\left\{\frac{2\epsilon+\beta}{d}+\frac{1}{2}\frac{\beta(2\epsilon+\beta)}{d^2}+\frac{1}{3}\frac{\epsilon^3+(\epsilon+\beta)^3}{d^3}+\cdots\right\}\end{array}\right]$$

$$= \frac{Rr}{2}\left[\frac{2}{d} + \frac{1}{3d^3}\left\{\frac{\epsilon^3+(\epsilon-\beta)^3}{2\epsilon-\beta} + \frac{\epsilon^3+(\epsilon+\beta)^3}{2\epsilon+\beta}\right\} + \frac{1}{4d^4}\left\{\frac{(\epsilon-\beta)^4-\epsilon^4}{2\epsilon-\beta} + \frac{(\epsilon+\beta)^4-\epsilon^4}{2\epsilon+\beta}\right\} + \frac{1}{5d^5}\left\{\frac{(\epsilon-\beta)^5+\epsilon^5}{2\epsilon-\beta} + \frac{(\epsilon+\beta)^5+\epsilon^5}{2\epsilon+\beta}\right\} \cdots\right]$$

$$= \frac{Rr}{2}\left[\frac{2}{d} + \frac{2}{3d^3}(\epsilon^2+\beta^2) + \frac{\epsilon\beta^2}{d^4} + \frac{2(\epsilon^4+4\epsilon^2\beta^2+\beta^4)}{5d^5} + \cdots\right] \qquad (4a)$$

But $C_0 = \frac{Rr}{d}$ is the capacity for concentric spheres,

$$\therefore \frac{\Delta_2 C}{C_0} = \tfrac{1}{3}\frac{\epsilon^2+\beta^2}{d^2} + \frac{\epsilon\beta^2}{2d^3} + \frac{\epsilon^4+4\epsilon^2\beta^2+\beta^4}{5d^5} + \cdots$$

If C_1 = capacity when the ball is in its central position, that is C_1 is the value of the expression (4*a*) when $\beta = 0$ (this is the same as expression 4), then

$$C - C_1 = \frac{Rr}{2}\left[\frac{2\beta^2}{3d^3} + \frac{\epsilon\beta^2}{d^4} + \frac{2\beta^2(4\epsilon^2+\beta^2)}{5d^5} + \cdots\right]$$

$$\therefore \frac{\Delta_2 C}{C_1} = \frac{\beta^2}{d^2}\left[\tfrac{1}{3} + \tfrac{1}{2}\frac{\epsilon}{d} + \frac{1}{45}\frac{31\epsilon^2+9\beta^2}{d^2} + \cdots\right]$$

This quantity $\frac{\Delta_2 C}{C_1}$ can be directly measured and in this way the validity of this method of treating the problem may be checked by experiment.

We know

$$2\epsilon = 0.017 \text{ cm}$$
$$d = 2.554 \text{ cm for ball A}$$
$$d = 3.798 \text{ cm for ball B}$$
$$\therefore \frac{\epsilon}{d} = 0.00332 \text{ cm for ball A}$$
$$= 0.00224 \text{ cm for ball B}$$

Hence

$$\frac{\Delta_2 C}{C_1} = \tfrac{1}{3}\frac{\beta^2}{d^2}\left[1 + 0.00498 + \ldots\right] \text{ for ball A}$$
$$= \tfrac{1}{3}\frac{\beta^2}{d^2}\left[1 + 0.00336 + \ldots\right] \text{ for ball B}$$

Whence we obtain the results given in Table X, which are represented by the curves on the accompanying plate, Fig. 11. The experimental values, which were obtained by displacing the ball

and observing the changes in the capacity, are marked on the same plate and it is evident that these two curves represent the facts within the limits of experimental error. The average difference between the calculated and observed values is only 2 or 3 parts in 100,000.

TABLE X.

Relative Changes in Capacity of Spherical Condenser due to Displacements β of Ball from Center.

Displacement β	Value of $\frac{\Delta_2 C}{C_1}$ for	
	Ball A	Ball B
0.01 cm	0.051×10^{-4}	0.023×10^{-4}
.02 "	0.205 "	0.093 "
.04 "	0.821 "	0.371 "
.06 "	1.849 "	0.835 "
.08 "	3.287 "	1.484 "
.10 "	5.135 "	2.319 "
.12 "	7.396 "	3.340 "
.16 "	13.147 "	5.936 "
.20 "	20.540 "	9.275 "
.24 "	29.582 "	13.356 "
.30 "	46.216 "	20.868 "
.36 "	66.560 "	30.050 "

When set up the balls are probably centered to within 0.1 mm, so that when first adjusted the capacity does not differ from that calculated by more than 1 in 100,000. Owing, however, to the slight warping of the stand, the stretching of the thread supporting the ball, etc., the ball may become excentric with the shell by possibly 0.2 mm; that is, the capacity may be in error by 2 in 100,000 for ball A; but even then for ball B it will be in error by less than 1 in 100,000. Wherever possible we have endeavored to determine this correction and to apply it. During the greater portion of the time a delicate level was kept on the arm carrying the vernier by which the vertical position of the ball was determined, and the leveling screws were adjusted as need be to keep this arm level. This

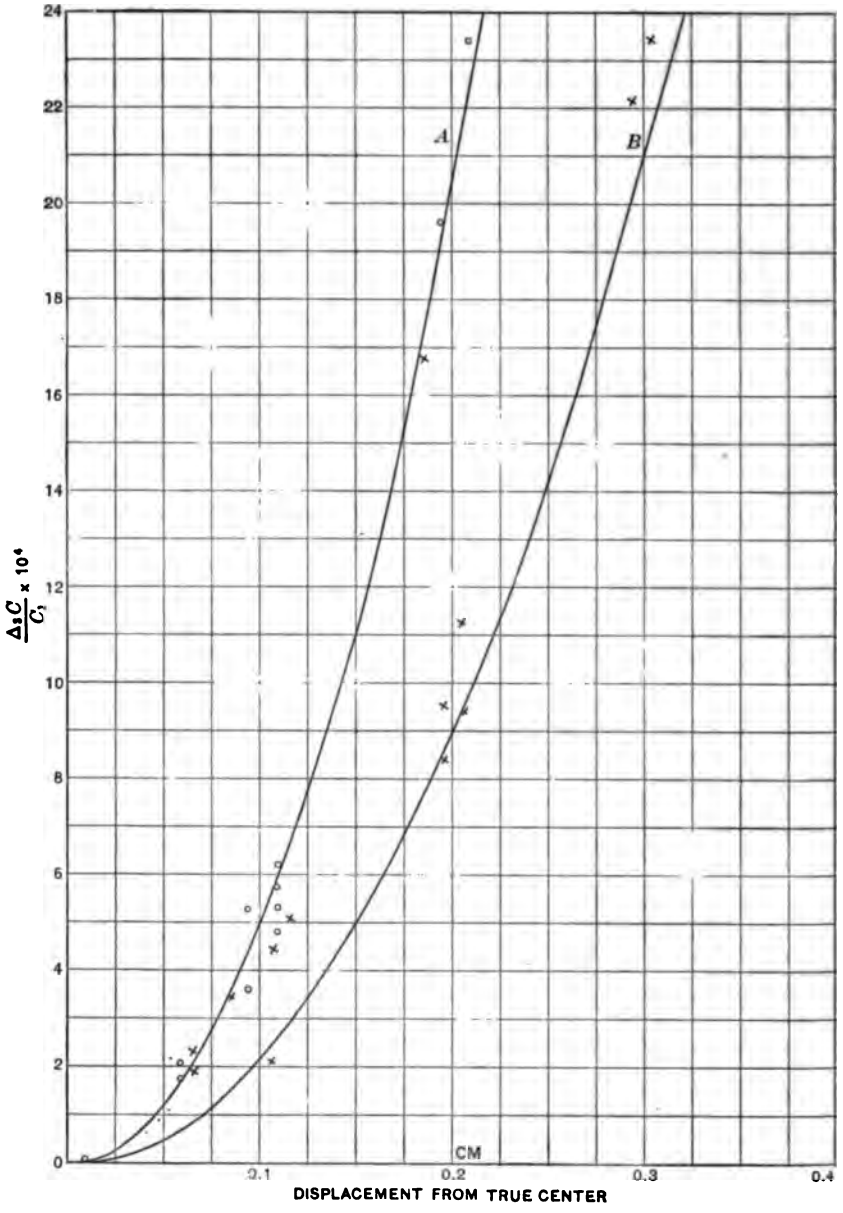

Fig. 11.—*Curve Showing the Change in the Capacity of the Spherical Condenser Due to a Displacement of the Inner Sphere.*

The upper curve applies to ball A; the lower to ball B.

largely eliminates displacement from the center due to the warping of the table and floor, but does not eliminate that due to a possible warping of the base of the instrument itself; this would tend to produce a very slight relative displacement of the shell and of the pillar from which the ball is suspended.

25. EFFECT OF HOLES IN SHELL, BUSHINGS, CORD, ETC.

There remains to be considered the effect of the holes in the shell, and of the ebonite bushings or tubes running through them, the small eye by which the silk cord is attached to the ball, and the effect of the silk cord.

The effect of the silk cord is surely less than that obtained by assuming that it is a rod passing radially from the ball to the shell (the cord does not touch the shell) without distorting the lines of force. Assuming its specific inductive capacity to be 2 and its diameter to be ½ mm, which is greater than the truth, we have for the relative increase in capacity

$$\frac{\delta C}{C} = \frac{\text{area of section of cord}}{\text{area of ball}} = \frac{\pi(0.025)^2}{4\pi(10.1)^2} = \frac{1}{653{,}000}$$
$$= 1.5 \text{ parts in } 1{,}000{,}000.$$

The leakage along the cord was found to be inappreciable.

The eye in the top of the ball is 2 mm in diameter and projects 2 mm above the surface of the ball; applying the method employed on page 474 we have for the capacity

$$C = \frac{Rr_1}{4\pi r_1^2}\left(4\pi r_1^2 - A + \frac{A}{d''}\right)$$

where r_1 is the *true radius* of the ball; d is the true radial distance from the surface of the ball to the shell; d'' is the distance from the top of the eye to the shell; A is the sectional area of the eye.

$$\therefore d'' = d - 0.2 \text{ cm}$$

$$A = 0.01\pi = 0.0314 \text{ cm}^2$$

In calculating the capacity we used the quantity r, which is the mean radius of the ball; that is, it is the radius of the perfect sphere having a volume equal to that of the ball plus the volume of the

eye, since the eye was in place when the volume was determined. Hence

$$\frac{4}{3}\pi r^3 = \frac{4\pi r_1^3}{3} + 0.2A$$

Putting $r_1 = r - \delta$ we find

$$\delta = \frac{0.2A}{4\pi r^2}$$

$$\therefore C = \frac{Rr_1}{d}\left\{1 - \frac{A}{4\pi r_1^2} + \frac{Ad}{4\pi r_1^2 d'}\right\} = \frac{Rr_1}{d}\left\{1 + \frac{\delta}{0.2}\,\frac{d-d'}{d'}\right\}$$

$$= \frac{Rr_1}{d}\left\{1 + \frac{\delta}{d'}\right\} \qquad (5)$$

The capacity we have assumed is

$$C_0 = \frac{Rr}{d_0}$$

where $d_0 = R - r = d - \delta$; substituting in (5) for r_1, d' and d their values in terms of r and d_0, we find for the true capacity, since $r_1 = r - \delta$, $d = d_0 + \delta$ and $d' = d_0 - 0.2 + \delta$,

$$C = \frac{Rr\left(1 - \frac{\delta}{r}\right)}{d_0\left(1 + \frac{\delta}{d_0}\right)}\left\{1 + \frac{\delta}{d_0\left(1 - \frac{0.2 - \delta}{d_0}\right)}\right\}$$

$$= \frac{Rr}{d_0}\left\{1 - \frac{\delta}{r} + \frac{\delta(0.2 - \delta)}{d_0^2}\right\}$$

$$= C_0\left(1 - \frac{\delta}{r} + \frac{0.2\delta}{d_0^2}\right) \text{ approximately.}$$

But $\delta = \frac{0.2A}{4\pi r^2}$ $\therefore \frac{\delta}{r} = \frac{0.2A}{4\pi r^3} = \frac{0.00628}{4\pi(10.12)^3}$ for ball A.

$$= 0.0000048$$

$$\frac{0.2\delta}{d_0^2} = 0.0000015$$

Hence the capacity we have assumed is too large by only 3.3 parts in 1,000,000. For ball B the correction is still less. Hence the correction for the eye is inappreciable.

The corrections for the ebonite were determined by measuring the change produced in the electromagnetic capacity when the ebonite was removed; and the effect of the side hole was determined by measuring the change in the electromagnetic capacity produced by plugging this hole with a brass rod. From this the effect of the top hole was calculated on the assumption that the relative effects are directly proportional to the areas of the two holes. This assumption was not tested experimentally, but is surely not far wrong.

Following is a summary of the various small corrections due to the holes and ebonite:

	Ball A	Ball B
Correction to capacity $\left(\frac{\Delta C}{C}\right)$ due to side hole	-11×10^{-6}	-8×10^{-6}
" " " " " " top "	-28×10^{-6}	-20×10^{-6}
" " " " " " side ebonite	$+8\times10^{-6}$	$+5\times10^{-6}$
" " " " " " top "	$+63\times10^{-6}$	$+32\times10^{-6}$
Total	$+32\times10^{-6}$	$+9\times10^{-6}$

Hence the total correction to be added to the calculated capacity is

	Ball A	Ball B
Correction $\left(\frac{\Delta C}{C}\right)$ for density of water	$+15\times10^{-6}$	$+15\times10^{-6}$
" " " error in shell	$+4\times10^{-6}$	$+2\times10^{-6}$
" " " holes and ebonite	$+32\times10^{-6}$	$+9\times10^{-6}$
" " " eye for suspending ball	-3×10^{-6}	-2×10^{-6}
Total	$=+48\times10^{-6}$	$+24\times10^{-6}$
Correction to **V**	$=+24\times10^{-6}$	$+12\times10^{-6}$
Correction to **V**, in centimeters per second	$=0.00007\times10^{10}$	$=0.00004\times10^{10}$

V. CORRECTIONS AND SOURCES OF ERROR—CYLINDRICAL CONDENSER.

The corrections and sources of error that must be considered in the case of the cylindrical condensers are as follows:

1. Error due to a lateral displacement of the axis of the inner cylinder.

2. Error due to an inclination of the axis of the inner cylinder with respect to the outer one.

3. Errors due to the figuring of the cylinders.

4. Correction for the presence of offsets where two sections come together.

These corrections and errors apply to the cylinders however they may be used. If they are used without guard cylinders we must also determine the two following corrections:

5. Correction due to the irregular distribution of the charge upon the ends of the inner cylinder and to the manner in which the magnitude of this charge is affected by slight variations in the dimensions of the ends of the different sections, and in particular the effect of varying the length of the condenser upon this charge.

6. Correction for possible variation in the capacity between the caps closing the ends of the cylinders.

If the cylinders are used with guard cylinders we must determine one other correction:

7. Correction for the gap between the guard cylinder and the middle section of the condenser.

26. CORRECTION FOR LATERAL DISPLACEMENT OF THE AXIS OF THE INNER CYLINDER.

We shall proceed as in the case of the spheres, page 477. The distance between the surfaces at any point (Fig. 12) is

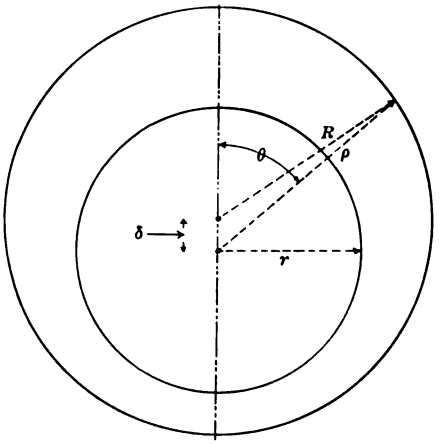

Fig. 12.

$$D_1 = \rho - r$$

$$\rho^2 + \delta^2 - 2\rho\delta\cos\theta = R^2$$

$$\therefore \rho\left\{1-\frac{\delta\cos\theta}{\rho}+\frac{\delta^2}{2\rho^2}-\frac{\delta^2\cos^2\theta}{2\rho^2}\cdots\right\}=R,\ \delta \text{ being very small.}$$

$$\therefore \rho=R+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2\rho}$$

$$=R+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2R},\ \text{approximately.}$$

$$\therefore D_1=R-r+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2R}$$

The distance for coaxial cylinders is

$$D_0=R-r$$

Hence the relative increase in the capacity of the system is

$$\frac{\Delta_1 C}{C_0}=\int_0^{2\pi}\frac{D_0-D_1}{D_0}\cdot\frac{d\theta}{2\pi}=\frac{\delta^2}{4R(R-r)} \qquad (6)$$

Hence the effect varies as the square of the displacement, and is negligible when δ is very small.

In building up the condensers the distance between the inner and the outer cylinders of each section was tested at their tops, at four points around the circumference, by means of a strip of brass cut wedge shaped with a slope of 1 mm in 10 cm, and the cylinders were slightly tipped by a suitable tightening of the screws fastening them to the lower section until they were as nearly coaxial as possible. The relative displacement of the axes at the top of any section was surely never over 0.025 mm. This determines the maximum error due to this cause.

Taking $R=7.25$
$r=6.25$
$\delta=0.0025$

we find from equation (6), $\frac{\Delta_1 C}{C_0}=0.22\times 10^{-6}$, which is negligible.

27. CORRECTION FOR INCLINATION OF THE AXIS OF THE INNER CYLINDER.

The error due to want of parallelism of the axes of the cylinders will be practically equal to that due to the same inclination of the two plates of a plane condenser, having the same length and distance apart, and of breadth πr, r being the radius of the inner cyl-

inder, end effects being neglected in both cases. We shall therefore calculate the increase in the capacity of a plane condenser when the plates are inclined at an angle θ. The lines of force passing from plate A at potential V to B at potential zero (neglecting effect of edges) will be arcs of circles. The electric force f will be uniform

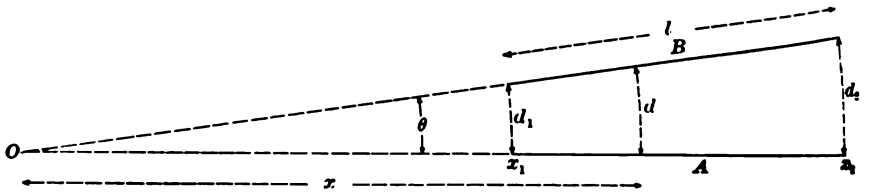

Fig. 13.

along any such line and equal to $V/\theta x$. If σ is the surface density, Q the total quantity of electricity on the plate of length l and breadth b, we have

$$f=\frac{V}{\theta x} \qquad \sigma=\frac{V}{4\pi\theta x} \qquad Q=b\int_{x_1}^{x_2}\sigma dx$$

$$\therefore\ Q=\frac{blV}{4\pi\theta}\int_{x_1}^{x_2}\frac{dx}{x}=\frac{blV}{4\pi\theta}\log\frac{x_2}{x_1}$$

The capacity then is

$$C_1=\frac{Q}{V}=\frac{b}{4\pi\theta}\log\frac{x_2}{x_1}=\frac{bl}{4\pi\delta}\log\frac{x_2}{x_1}$$

since

$$\theta=\frac{d_2}{x_2}=\frac{d_1}{x_1}=\frac{d_2-d_1}{x_2-x_1}=\frac{\delta}{l}$$

Also, since

$$\frac{x_2}{x_1}=\frac{d_2}{d_1}$$

$$C_1=\frac{bl}{4\pi\delta}\log\left(1+\frac{\delta}{d_1}\right)$$

$$=\frac{bl}{4\pi d_1}\left(1-\frac{1}{2}\frac{\delta}{d_1}+\frac{1}{3}\frac{\delta^2}{d_1^2}-\cdots\right)$$

If d_1 is the mean distance between the plates, the point x_1 will

correspond to the middle point of the plate, and the expression just obtained will be the capacity of that half of the plate for which the distance between the planes is greater than d_1. For the other half of the plate, $\log \frac{x_2}{x_1}$ becomes $\log \frac{d_1}{d_1-\delta} = -\log\left(1-\frac{\delta}{d_1}\right)$ and the expression for the capacity of this half is

$$C_2 = \frac{bl}{4\pi d_1}\left(1+\frac{1}{2}\frac{\delta}{d_1}+\frac{1}{3}\frac{\delta^2}{d_1^2}+\ldots\right)$$

The sum of these, which is the capacity of the entire condenser, is

$$C = C_1+C_2 = \frac{2bl}{4\pi d_1}\left(1+\frac{1}{3}\frac{\delta^2}{d_1^2}+\frac{1}{5}\frac{\delta^4}{d_1^4}+\ldots\right) \qquad (7)$$

But, if the plates were parallel and at the same mean distance, their capacity would be

$$C_0 = \frac{2bl}{4\pi d_1}$$

$$\therefore \frac{\Delta_2 C}{C_0} = \frac{C-C_0}{C_0} = \frac{1}{3}\frac{\delta^2}{d_1^2}+\frac{1}{5}\frac{\delta^4}{d_1^4}+\ldots. \qquad (8)$$

Hence the increase in the capacity of the coaxial cylinders due to a slight tipping of the inner with respect to the outer can not be greater than the quantity given in (8), which is obviously negligible when δ/d_1 is very small. If δ/d_1 is 0.01, $\frac{\Delta_2 C}{C_0}$ amounts to about 1 part in 30,000.

By the method of adjustment described on page 485 it is evident that the greatest possible tipping will occur when the bottom of a section is displaced 0.025 mm in one direction and the top an equal amount in the other direction. This would correspond to $\delta =$ 0.0025 cm. Substituting this in the expression for $\frac{\Delta_2 C}{C_0}$ for these cylinders, we find that the maximum error due to the relative tipping of the axis of the cylinders can not exceed about 2 in 1,000,000. Usually the tipping was much less than that here assumed.

28. CORRECTION FOR ERRORS IN FIGURING.

These errors are due (1) to conicality of the cylinders, (2) to ellipticity, (3) to other irregularities. The sections of all the cylinders

are quite accurately circular, and the smaller irregularities in the surface are too slight to cause an error as large as 1 part in 100,000. The outer cylinders are, however, slightly variable in section as we pass along their length, so that there is an appreciable conicality, the effect of which needs to be investigated.

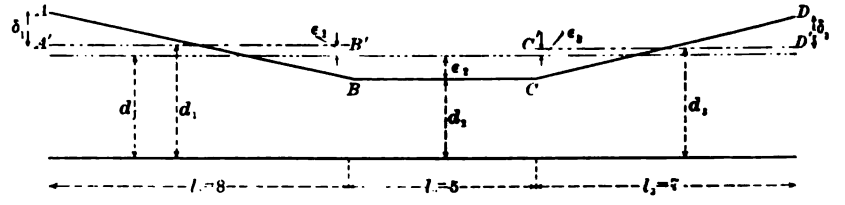

Fig. 14.

If d_1 is the mean distance between the cylinders for the first section, and δ_1 is the maximum variation of the distance from this mean for this conical portion, we can calculate the difference in the capacity between this conical section and one of uniform distance d_1 by the formula of the preceding section. This may be done for each section. Then we can calculate the difference in capacity between the several cylindrical sections of distances d_1, d_2, d_3, etc., and the single cylinder having the same mean distance d. The sum of these differences will be the correction sought. Let us take a parallel plate condenser that is made up of three sections, two of which, AB and CD, are not parallel. $AA'=\delta_1$, $DD'=\delta_3$. The parallel plates A'B' and C'D', which are nearly equivalent to AB and CD, are distant $d+\epsilon_1$, and $d+\epsilon_3$; and BC is distant $d-\epsilon_2$ from the lower plate.

The first section of distance $d_1=d+\epsilon_1$ has a capacity

$$C_1=\frac{A_1}{4\pi d_1}=\frac{bl_1}{4\pi}\cdot\frac{1}{d+\epsilon_1}=\frac{bl_1}{4\pi d}\left(1+\frac{\epsilon_1}{d}\right)^{-1}$$

$$\text{or, } C_1=\frac{bl_1}{4\pi d}\left\{1-\frac{\epsilon_1}{d}+\frac{\epsilon_1^2}{d^2}-\ \ldots\right\}$$

$$\text{Similarly, } C_2=\frac{bl_2}{4\pi d}\left\{1+\frac{\epsilon_2}{d}+\frac{\epsilon_2^2}{d^2}+\ \ldots\right\}\text{assuming } \epsilon_2 \text{ is negative.}$$

$$C_3=\frac{bl_3}{4\pi d}\left\{1-\frac{\epsilon_3}{d}+\frac{\epsilon_3^2}{d^2}-\ \ldots\right\}$$

Adding these equations for the total capacity of the three sections

we have, since the total area A is $bl_1+bl_2+bl_3$ and $l_1\epsilon_1+l_3\epsilon_3=l_2\epsilon_2$

$$C=\frac{A}{4\pi d}\left\{1+\frac{l_1\epsilon_1^2+l_2\epsilon_2^2+l_3\epsilon_3^2}{ld^2}+\ldots\right\}$$

$$\therefore \frac{\Delta_3 C}{C_0}=\frac{l_1}{l}\frac{\epsilon_1^2}{d^2}+\frac{l_2}{l}\frac{\epsilon_2^2}{d^2}+\frac{l_3}{l}\frac{\epsilon_3^2}{d^2}$$

Thus, the increase in capacity is the weighted mean of $\left(\frac{\epsilon}{d}\right)^2$ for all the sections, while the difference between the conical sections and the corresponding parallel ones is $\frac{1}{3}$ the weighted mean of $\left(\frac{\delta}{d}\right)^2$

As an example, suppose $l_1=8$, $l_2=5$, $l_3=7$ cm

Take $\epsilon_1=+0.01$ mm $\quad\delta_1=0.054$ mm $\quad d=1$ cm

$\epsilon_2=-0.044$ " $\quad\delta_2=0$

$\epsilon_3=+0.02$ " $\quad\delta_3=0.064$ mm

These are small differences, but very appreciable, being of the same order of magnitude as those of our cylinders, as shown by Tables III, IV, V, and VI.

$\Sigma\left(\frac{\epsilon}{d}\right)^2$ will here be 0.0000066

$\frac{1}{3}\Sigma\left(\frac{\delta}{d}\right)^2$ " " " 0.0000087

Sum $=0.0000153=1.5$ parts in 100,000.

Applying this method to the cases of the outer cylinders Nos. 2 and 3, we find:

Correction for No. $2=+1.0$ part in 100,000

" " No. $3=+1.8$ " " 100,000.

The mean of these is 1.4 parts in 100,000 in the electrostatic capacity, the latter being larger than if the cylinders were perfectly cylindrical. These cylinders could be reground with our present facilities so that the correction due to conicality would be not more than one-tenth as great, and hence wholly negligible in the most refined work.

29. CORRECTION FOR OFFSETS.

In the treatment of this and several succeeding problems we shall, as a first approximation, assume that we may consider each element of the cylindrical condenser as an element of a parallel plate condenser. The problem of obtaining the distribution of charge along the length of the condenser then becomes a two-dimensional problem and may be solved by the method employed by J. J. Thomson in the third chapter of "Recent Researches." The method consists in transforming the polygon defined by the section of the condenser by a plane through its axis (call this the z plane) into the real axis of what we shall call the t plane. The polygon, defined by this section, in the w plane ($w=\phi+i\psi$, where ϕ is the stream function and ψ is the potential function) is also to be transformed into the real axis of the t plane and made to correspond there point by point with the other polygon. By eliminating t from these two equations of transformation we get ϕ and ψ as function of x and y, $(x+iy=z)$, and so can obtain the distribution.

The transformation to be employed is

$$\frac{dz}{dt}=C\left(t-t_1\right)^{\frac{a_1}{\pi}-1}\left(t-t_2\right)^{\frac{a_2}{\pi}-1}\left(t-t_3\right)^{\frac{a_3}{\pi}-1}\ldots$$

where a_1, a_2, a_3 . . . . are the interior angles of the polygon at the vertices corresponding to $t=t_1$, $=t_2$, $=t_3$ . . .

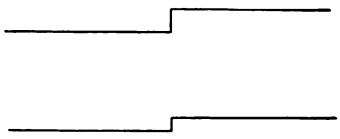

Fig. 15.

The general problem we desire to solve is that of two parallel broken planes, the two steps being in the same normal plane and of unequal height (Fig. 15). While this problem can be solved, it is easier to consider another problem, viz, the problem in which there is a step in but one of the planes; from this, by the method of images, we at once get the solution of the problem in which there are two equal steps. From these two problems we can obtain with sufficient accuracy the effect of the second step.

A condenser consisting of an infinite plane parallel to two semi-infinite planes joined by a vertical step.—The diagram in the z plane is as shown in Fig. 16. Let p and q be the distances of the infinite plane from the two semi-infinite planes. Then $q-p$ will be the height of the step.

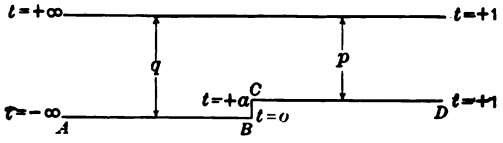

Fig. 16.

Take $t=+\infty$ at $x=-\infty$ on the infinite plate,

$t=+1$ at $x=+\infty$ on the infinite plate,

$t=0$ at B, the foot of the step; B is taken as the origin. Then at the top of the step, t will have some value a, between 0 and $+1$, to be determined from the dimensions of the apparatus. The transformation is

$$\frac{dz}{dt}=\frac{C}{t-1}\sqrt{\frac{t-a}{t}}$$

$$\therefore z=C\left[\log\left(\sqrt{t(t-a)}+t-\frac{a}{2}\right)-\sqrt{1-a}\,\log\left(\frac{\sqrt{t(t-a)}+\sqrt{1-a}}{t-1}+\frac{2-a}{2\sqrt{1-a}}\right)\right]$$

$$\therefore z=-\frac{q}{\pi}\left[\log\frac{2\sqrt{t(t-a)}+2t-a}{a}-\sqrt{1-a}\,\log\frac{2\sqrt{1-a}\sqrt{t(t-a)}-a+t(2-a)}{a(t-1)}\right]+iq$$

if the origin is taken at the point $t=0$. But, as t decreases through $+1$, z must decrease by ip,

$$\therefore\sqrt{1-a}=\frac{p}{q}$$

$$\therefore z = -\frac{q}{\pi}\log\frac{2q\sqrt{t[tq^2-(q^2-p^2)]}-(q^2-p^2)+2tq^2}{q^2-p^2}$$

$$+\frac{p}{\pi}\log\frac{2p\sqrt{t[tq^2-(q^2-p^2)]}-(q^2-p^2)+(q^2+p^2)t}{(q^2-p^2)(t-1)}+iq$$

The diagram in the w plane consists of two infinite lines (Fig. 17), and the transformation is

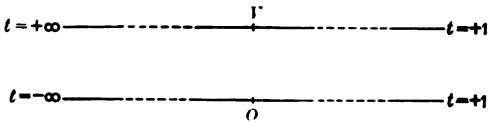

Fig. 17.

$$\frac{dw}{dt} = \frac{B}{t-1}$$

$$\therefore w = -\frac{V}{\pi}[\log(t-1) - i\pi]$$

if the potential of the upper plane is V and that of the lower one is zero. Hence the charge on a unit strip bounded by t_1 and t_2, $t_1 < t_2$, is

$$\sigma = -\frac{1}{4\pi}\frac{\partial\psi}{\partial\nu} = -\frac{1}{4\pi}\frac{\partial\phi}{\partial s}$$

$$E = \int_{s_1}^{s_2}\sigma ds = -\frac{1}{4\pi}[\phi(t_2)-\phi(t_1)]$$

$$\therefore E = \frac{V}{4\pi^2}\log\frac{t_2-1}{t_1-1}$$

Hence, the charge on the semi-infinite plane at the top of the step, and extending from $t=a$ to $t=t_1$ (t_1 nearly $=1$) is

$$E_1 = \frac{V}{4\pi^2}\log\frac{1-t_1}{1-a}$$

But, for t nearly equal to but less than 1, we have

$$w_1 = -\frac{q}{\pi}\log\frac{q+p}{q-p} + \frac{p}{\pi}\log\frac{4p^2}{q^2-p^2} - \frac{p}{\pi}\log(1-t_1)$$

$$\therefore E_1 = -\frac{V}{4\pi p}\left[x_1 + \frac{q}{\pi}\log\frac{q+p}{q-p} - \frac{p}{\pi}\log\frac{4q^2}{q^2-p^2}\right]$$

If q differs but slightly from p, this expression can be put in a form more suitable for computation by putting

$$q = p\,(1+a)$$

Then,

$$\frac{q+p}{q-p} = \frac{2+a}{a}$$

$$\frac{4q^2}{q^2-p^2} = \frac{4(1+a)^2}{a(2+a)}$$

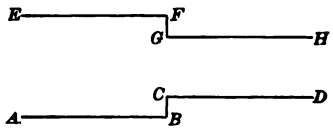

Fig. 18.

$$\therefore E_1 = -\frac{V}{4\pi p}\left[x_1 + \frac{p}{\pi}\left\{\log\frac{2+a}{a}\cdot\frac{a(2+a)}{4(1+a)^2} + a\log\frac{2+a}{a}\right\}\right]$$

$$= -\frac{V}{4\pi p}\left[x_1 + \frac{p}{\pi}\left\{a\log\frac{2+a}{a} - \log\left(1+a-\frac{a^2}{4}\right)\right\}\right]$$

$$= -\frac{V}{4\pi p}\left[x_1 + \frac{pa}{\pi}\left\{\log\frac{2+a}{a} - 1 + \frac{3a}{4}\ldots\right\}\right]$$

By the method of images, we find that the corresponding expression for the case of two equal steps, as shown in Fig. 18, is

$$E_1' = -\frac{V}{4\pi p}\left[x_1 + \frac{pa}{\pi}\left\{\log\frac{(1+a)}{a} - 1 + \frac{3a}{2}\right\}\right]$$

in which p (the distance between GH and CD) and the height of each step (ap) have the same numerical values as before:

$$\therefore E_1 - E_1' = -\frac{V}{4\pi p}\cdot\frac{pa}{\pi}\left\{\log\frac{2+a}{a} - \log\frac{1+a}{a} - \frac{3a}{4}\right\}$$

$$= -\frac{Va}{4\pi^2}\left\{\log\left(2-\frac{a}{1+a}\right) - \frac{3a}{4}\right\}$$

$$= -\frac{Va}{4\pi^2}\left\{\log 2 - \frac{5a}{4} + \ldots\right\}$$

$$= -\frac{Va}{4\pi}(0.21) \text{ approximately.} \qquad (9)$$

Hence the effect of moving the portion EF of the plane away from AB by an amount ap is to decrease the effective length of the portion CD of the other plane by 0.21 ap. If it had been moved in the other direction the effective length would have been increased by the same absolute amount. Hence if there is a step of height $a_1 p$ in the upper plane and one of height ap in the lower one (the planes of the steps coinciding) the charge on the section CD is very nearly equal to that on a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section CD and of length greater than that of CD by an amount

$$\delta l_1 = \frac{pa}{\pi}\left\{\log\frac{2+a}{a} - 1 + \frac{3a}{4}\right\} \mp \frac{pa_1}{\pi}\left(\log 2 - \frac{5a_1}{4}\right)$$

The minus sign is to be taken if the relative positions of the steps is as shown in Fig. 18.

The charge on the side of the step, between $t=o$ and $t=a$, is

$$E_2 = \frac{V}{4\pi^2}\log(1-a) = -\frac{V}{4\pi^2}\log\frac{q^2}{p^2}$$

To reduce this to a form suitable for computation, when q is nearly equal to p, put $q=p(1+a)$, and we find

$$\begin{aligned} E_2 &= -\frac{V}{4\pi^2}\log(1+a)^2 \\ &= -\frac{V}{2\pi^2}\left\{a - \frac{a^2}{2} + \ldots\right\} \end{aligned} \tag{10}$$

By the method of images we find for the case of two equal steps, as shown in Fig. 18, the expression

$$E'_2 = -\frac{V}{4\pi^2}\log\frac{q_1}{p_1}$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2}\log\frac{q^2}{p^2}\cdot\frac{p_1}{q_1}$$

If we regard p as remaining fixed, then

$$\begin{aligned} p_1 &= p \\ q_1 &= q + ap = p(1+2a) \end{aligned}$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2} \log \frac{(1+a)^2}{1+2a}$$

$$= -\frac{V}{4\pi^2}(a^2 - 2a^3 + \ldots)$$

Hence the effect of moving EF away from AB by an amount ap is to decrease the charge on the side of the step by the amount, $\frac{Ia^2}{4\pi^2}$, approximately. Similarly, if we regard q as remaining fixed, we must take

$$q_1 = q = p(1+a)$$
$$p_1 = p(1-a)$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2} \log (1-a^2)$$

$$= +\frac{V}{4\pi^2}\left(a^2 + \frac{a^4}{2} + \ldots\right)$$

Therefore, the effect of moving GH toward CD by an amount ap is to increase the charge on the side of the step by an amount $\frac{Va^2}{4\pi^2}$, approximately. It is readily seen that these effects are exactly equal to half the effect produced by moving the entire upper plane by the same amount. Hence the addition of a step equal to βp in the upper plane will increase the charge on the side of the step in the lower plane by $\frac{\beta a V}{4\pi^2} = \frac{V}{4\pi}(0.32a\beta)$. Since this effect varies as the product of a and β, it is very small as compared with the effects given by expressions (9) and (10), if a and β are small.

The charge on the semi-infinite plane at the bottom of the step between, $t = o$ and $t = -\tau$, where τ is very large, is

$$E_3 = \frac{V}{4\pi^2} \log\left(\frac{1}{1+\tau}\right)$$
$$= -\frac{V}{4\pi^2} \log \tau, \text{ approximately.}$$

But for $t = -\tau$, τ being large, we have

$$x_{-\infty} = -\frac{q}{\pi}\log\tau + \frac{q}{\pi}\log\frac{q^2-p^2}{4q^2} + \frac{p}{\pi}\log\frac{q+p}{q-p}$$

$$\therefore E_3 = -\frac{V}{4\pi q}\left[X - \frac{q}{\pi}\log\frac{4q^2}{q^2-p^2} + \frac{p}{\pi}\log\frac{q+p}{q-p}\right]$$

where $X = -x_{-\infty}$. That is, X is the distance from B. To reduce this expression to a form more suitable for computation when q is nearly equal to p, put

$$p = q\,(1-\beta)$$

Then

$$E_3 = -\frac{V}{4\pi q}\left[X - \frac{q}{\pi}\log\frac{4}{\beta(2-\beta)} + \frac{q(1-\beta)}{\pi}\log\frac{2-\beta}{\beta}\right]$$

$$= -\frac{V}{4\pi q}\left[X - \frac{q\beta}{\pi}\left\{1 + \log\frac{2-\beta}{\beta} + \frac{\beta}{4} + \cdots\right\}\right]$$

Applying the method of images, we find for the case of equal steps, as shown in Fig. 18,

$$E_3' = -\frac{V'}{4\pi q'}\left[X - \frac{q'\beta'}{\pi}\left\{1 + \log\frac{2-\beta'}{\beta'} + \frac{\beta'}{4}\cdots\right\}\right]$$

where p' and q' are the half distances between the plates, V' is half the potential difference, and $p' = q'\,(1-\beta')$. If we take $2q' = q$, and $p' = p - \beta q$, we find

$$\beta' = 2\beta$$

$$\therefore E_3' = -\frac{V}{4\pi q}\left[X - \frac{q\beta}{\pi}\left\{1 + \log\frac{1-\beta}{\beta} + \frac{\beta}{2} + \cdots\right\}\right]$$

$$\therefore E_3 - E_3' = +\frac{V\beta}{4\pi^2}\left\{\log\frac{2-\beta}{1-\beta} - \frac{\beta}{4}\right\}$$

$$= +\frac{V\beta}{4\pi^2}\left\{\log 2 + \frac{\beta}{4} + \frac{3\beta^2}{8} + \cdots\right\}$$

Hence the effect of moving the portion GH of the plane nearer to CD by an amount βq is to increase the effective length of the portion AB by an amount

$$\frac{\beta q}{\pi}\left\{\log 2 + \frac{\beta}{4} + \cdots\right\}$$

Which for small values of β becomes, approximately, 0.21 βq. Hence if there is a step of $\beta_1 q$ in the upper plane and one of βq in the lower one—the planes of the steps coinciding—the charge on the section AB is very nearly equal to that of a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section AB and of a length greater than that of AB by an amount

$$\delta l_3 = -\frac{q\beta}{\pi}\left\{1+\log\frac{2-\beta}{\beta}+\frac{\beta}{4}\right\} \pm\frac{q\beta_1}{\pi}\left\{\log 2+\frac{\beta_1}{4}+\ldots\right\}$$

The plus sign is to be taken if the relative position of the steps is as shown in Fig. 18.

If there is no offset in the upper plate, the total charge on the plate AD is

$$E=-\frac{V}{4\pi}\left[\begin{array}{rl} \frac{x_1}{p}+\frac{a}{\pi}\Big\{ & \log\frac{2+a}{a}-1+\frac{3a}{4}+\ldots\Big\} \\ +\frac{a}{\pi}\Big\{ & 2-a+\ldots\Big\} \\ +\frac{X}{q}-\frac{\beta}{\pi}\Big\{ & \log\frac{2-\beta}{\beta}+1-\frac{\beta}{4}+\ldots\Big\} \end{array}\right]$$

But in this case

$$1-\beta=\frac{1}{1+a}$$

$$\therefore\beta=\frac{a}{1+a}$$

$$\therefore E=-\frac{V}{4\pi}\left[\begin{array}{rl} \frac{x_1}{p}+\frac{a}{\pi}\Big\{ & \log\frac{2+a}{a}-1+\frac{3a}{4}\ldots\Big\} \\ +\frac{a}{\pi}\Big\{ & 2-a\ldots\Big\} \\ +\frac{X}{q}-\frac{a}{\pi}\Big\{ & \frac{1}{1+a}\log\frac{2+a}{a}+1-\frac{5a}{4}\ldots\Big\} \end{array}\right]$$

$$=-\frac{V}{4\pi}\left[\frac{x_1}{p}+\frac{X}{q}+\frac{a}{\pi}\left\{\frac{a}{1+a}\log\frac{2+a}{a}+a+\ldots\right\}\right]$$

Hence the effect of a step in one of a pair of parallel plates is to

increase the magnitude of the charge by

$$\delta E = \frac{Va^2}{4\pi^2}\left\{1 + \frac{1}{1+a}\log\frac{2+a}{a} + \ldots\right\}$$

Let us now determine the distance to which the effect of the step makes itself felt. To do this, differentiate the charge with respect to the distance to get the density, and determine the distance we must go from the step before the density becomes sensibly equal to its value at infinity.

Side CD at the top of the step:

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial t}\cdot\frac{\partial t}{\partial x} = +\frac{V}{4\pi^2}\cdot\frac{1}{1-t_1}\cdot\frac{(t_1-1)\pi}{q}\sqrt{\frac{t_1}{t_1-a}}$$

$$= -\frac{V}{4\pi q}\sqrt{\frac{t_1}{t_1-a}} \equiv \sigma_1$$

Put $t_1 = 1-\epsilon$, then

$$\sigma_1 = -\frac{V}{4\pi q}\sqrt{\frac{1-\epsilon}{1-a-\epsilon}}$$

$$= -\frac{V}{4\pi q}\cdot\frac{1}{\sqrt{1-a}}\sqrt{\frac{1-\epsilon}{1+\frac{\epsilon}{1-a}}}$$

$$= -\frac{V}{4\pi q}\cdot\frac{1}{\sqrt{1-a}}\left\{1 + \frac{a\epsilon}{2(1-a)}\right\}$$

$$= -\frac{V}{4\pi p}\left\{1 + \tfrac{1}{2}\frac{a\epsilon}{1-a}\right\}$$

The density at $x = \infty$ is

$$\sigma_0 = -\frac{V}{4\pi p}$$

$$\therefore \frac{\sigma_1 - \sigma_0}{\sigma_0} = \frac{a\epsilon}{2(1-a)}$$

If this ratio is equal to a very small quantity, say η, then

$$\epsilon = \frac{2(1-a)\eta}{a}$$

or $$\epsilon = \frac{2p^2\eta}{q^2-p^2}$$

Hence, the distance at which $\frac{\sigma_1-\sigma_2}{\sigma_0} = \eta$ must sensibly be

$$x_1 = -\frac{q}{\pi}\log\frac{q+p}{q-p} + \frac{p}{\pi}\log\frac{4p^2}{q^2-p^2} - \frac{p}{\pi}\log\frac{2p^2\eta}{q^2-p^2}$$

$$= -\frac{q}{\pi}\log\frac{q+p}{q-p} + \frac{p}{\pi}\log\frac{2}{\eta}$$

Example:

If $q = 1.0000$ cm, $q-p = 0.006$ cm,
then,

$$x_1 = \frac{0.994}{\pi}\log\frac{1}{\eta} - 1.629$$

For $\eta = 10^{-5}$ this becomes

$$x_1 = 2.01 \text{ cm}$$

Hence, for such a step, the density on the plane at the top of the step and at a distance from it greater than 2 cm does not differ from its value at an infinite distance by more than 1 in 100,000.

Side AB at the bottom of the step:

$$\frac{\partial E}{\partial \tau}\cdot\frac{\partial \tau}{\partial x} = -\frac{V}{4\pi^2}\cdot\frac{1+\tau}{(1+\tau)^2}\cdot\frac{\pi(1+\tau)}{q}\sqrt{\frac{\tau}{a+\tau}}$$

$$= -\frac{I}{4\pi q}\sqrt{\frac{\tau}{a+\tau}} \equiv \sigma_3$$

Here, $$\sigma_0 = -\frac{I}{4\pi q}$$

$$\therefore \frac{\sigma_3-\sigma_0}{\sigma_0} = -1+\sqrt{\frac{\tau}{a+\tau}}$$

Since τ is very large we may write

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = -1+\left(1+\frac{a}{\tau}\right)^{-\frac{1}{2}} = +\frac{1}{2}\frac{a}{\tau}$$

If this ratio is to be a very small quantity, say η, then

$$\frac{1}{2}\frac{a}{\tau}=\eta$$

$$\text{or, } \tau=\frac{1}{2}\frac{a}{\eta}$$

Hence, the distance at which $\frac{\sigma_s-\sigma_0}{\sigma_0}=\eta$ must sensibly be

$$X\equiv -x_{-\infty}=\frac{q}{\pi}\log\frac{a}{2\eta}-\frac{q}{\pi}\log\frac{q^2-p^2}{4q^2}-\frac{p}{\pi}\log\frac{q+p}{q-p}$$

$$=\frac{q}{\pi}\log\frac{2}{\eta}-\frac{p}{\pi}\log\frac{q+p}{q-p}$$

Example:

$$\text{If } q=1.000 \text{ cm; } q-p=0.006 \text{ cm}$$

then,

$$X=0.318\log\frac{1}{\eta}-1.398$$

For $\eta=10^{-5}$ this becomes

$$X=2.26 \text{ cm}$$

Let us now apply these equations to the cases that arise in the work with the cylinders. Denoting by h the distance between the opposed surfaces at the end of the sections; by δ the excess of the radius of the upper section of the inner cylinder above that of the lower one, measured at the adjoining ends, and by δ_1, the corresponding quantity for the outer cylinders, we obtain from our measurements the following table:

TABLE XI.

Junction of sections	h		δ	δ_1
	Upper	Lower		
1 and 2	0.98793	0.98434	0.00074	0.00433
1 and 3	0.98351	0.98434	0.00016	0.00067
2 and 3	0.98351	0.98559	−0.00011	−0.00219
2 and 4	0.98099	0.98559	−0.00110	−0.00570
3 and 4	0.98099	0.98604	−0.00107	−0.00612

The greatest correction term will evidently be that which arises from the offsets at the junction of sections 2 and 4. Taking p as the distance between the surfaces at the upper end of section 2, we have:

$$p = 0.98559$$
$$q = 0.98099$$
$$a = 0.00111_5$$
$$\beta = 0.00111_9$$
$$a_1 = 0.00578_2$$
$$\beta_1 = 0.00580_8$$

Hence, the charge on the inner cylinder of section No. 2—neglecting the face of the step—is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by an amount

$$\frac{V}{4\pi p}\Delta l_1 = \frac{V}{4\pi_2}\left[a\left\{\log\frac{2+a}{a} - 1 + \frac{3a}{4}\right\} + a_1\left\{\log 2 - \frac{5a_1}{4}\right\}\right] = 0.011121\frac{V}{4\pi^2}$$

Similarly, the charge on the inner cylinder of section No. 4 is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by

$$\frac{V}{4\pi q}\Delta l_3 = -\frac{V}{4\pi^2}\left[\beta\left\{\log\frac{2-\beta}{\beta} + 1 + \frac{\beta}{4}\right\} + \beta_1\left\{\log 2 + \frac{\beta_1}{4}\right\}\right] = -0.01353\frac{V}{4\pi^2}$$

The charge on the side of the step is

$$\frac{V}{4\pi^2}\left[2a\left(1 - \frac{a}{2}\right) + aa_1\right] = 0.002235\frac{V}{4\pi^2}$$

Hence the net effect of these offsets is to increase the total charge on the inner cylinder by

$$-0.00008\frac{V}{4\pi^2}$$

which is equivalent to decreasing the length of the condenser by only 0.2μ, a quantity entirely negligible.

Hence the only case in which the presence of the offsets can produce an appreciable effect is when they occur at the gap separating the guard cylinder from the condenser proper, under which condition the charge corresponding to one of the Δl correction terms is not measured.

30. CORRECTION DUE TO IRREGULAR DISTRIBUTION OF CHARGE.

The determination of the various corrections due (1) to the irregular distribution of charge near the end of the inner cylinder, (2) to the charge on the interior of the inner cylinder and (3) to the variation of the charge on the outside of the outer cylinder as the latter is changed in height, are all parts of the same mathematical problem. Being unable to treat directly this problem for the case of circular cylinders, we shall solve two cases for systems of semi-infinite plates, viz, (1) the case of four infinitely thin semi-infinite plates (this will give us an idea of the distribution on the interior of the inner cylinder); (2) the case of a thick semi-infinite plate midway between two infinitely thin semi-infinite plates (this will give us a minimum value for the charge on the end of the inner cylinder when the latter is closed). A more interesting case is that in which the outer plates

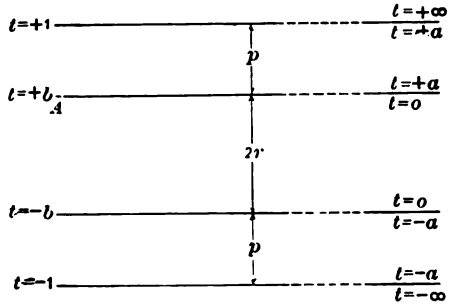

Fig. 19.

are of finite thickness, but the increased mathematical difficulties more than offset the gain.

Case 1. *A condenser consisting of two mutually exterior, identical pairs of semi-infinite planes, all bounded by straight lines lying in a plane normal to the four.*—The innermost planes are both at zero potential, while the outermost ones are at the common potential V. Assume that the origin lies at the point A, Fig. 19, the y axis being normal to the planes. Assume $t=+\infty$ at the point $x=\infty$ on the upper side of the top plane; $t=+a$ at the corresponding point on the lower side; $t=+1$ at the bounding edge of the top plane; $t=+b$ at the bounding edge of the next to the top plane; $t=+a$ on the top face of the next to the top plane at $x=\infty$; $t=0$ at the corresponding point on the lower side of this plane. At the

corresponding points of the other pair of planes t will have the same absolute values, but will be of the opposite sign. The values of a and b must be determined from the distances between the planes. Let p be the distance between the planes composing either pair, and $2r$ be the distance between the two innermost planes.

The transformation is

$$\frac{dz}{dt} = C\frac{(t^2-1)(t^2-b^2)}{t(t^2-a^2)}$$

$$\text{or,} \quad z = -\frac{C}{2a^2}\Big[b^2 \log\frac{t^2}{b^2} + (1-a^2)(a^2-b^2)\log\frac{t^2-a^2}{a^2-b^2} - a^2(t^2-b^2) - i\pi(1-a^2)(a^2-b^2)\Big]$$

The conditions that z shall increase by $2ir$ as t increases through zero, and by ip as t increases through a give us the equations

$$C = \frac{2ra^2}{\pi b^2} = \frac{2pa^2}{\pi(1-a^2)(a^2-b^2)}$$

These, combined with the condition that $x=o$ for $t=+1$, give the equation

$$\frac{ra^2(1-b^2)}{b^2} = 2r\log\frac{1}{b} + p\log\frac{1-a^2}{a^2-b^2} \tag{11}$$

From the expression for C, we find

$$b^2 = \frac{a^2r(1-a^2)}{p+r-a^2r} \tag{12}$$

The last two equations allow us to determine the values of a and of b. Replacing C by its value as given above, we find

$$z = -\frac{2r}{\pi}\log\frac{t}{b} - \frac{p}{\pi}\log\frac{t^2-a^2}{a^2-b^2} + \frac{ra^2}{\pi b^2}(t^2-b^2) + ip \tag{13}$$

The diagram in the w plane consists of two infinite parallel lines; the values of t at the extremities of these lines are $+a$ and $-a$. The transformation is

$$\frac{dw}{dt} = B\frac{1}{t^2-a^2}$$

$$\therefore w=\frac{V}{\pi}\left[\log\frac{t+a}{t-a}+i\pi\right]$$

Hence, the charge per unit length of the strip bounded by $t=t_1$ and $t=t_2$ is

$$E=\frac{V}{4\pi^2}\log\left[\frac{t_2-a}{t_2+a}\cdot\frac{t_1+a}{t_1-a}\right],\ t_1<t_2$$

The charge on the top face of the top plate is obtained by putting $t_1=1$; $t_2=t$, where t is very large. Then

$$E_1=\frac{V}{4\pi^2}\log\left[\frac{t-a}{t+a}\cdot\frac{1+a}{1-a}\right] \tag{14}$$

$$=\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a}{t}\right]$$

But, for t very large

$$t=\frac{b^2}{ra}\left[\pi x-2r\log b-p\log(a^2-b^2)\right]$$

$$\therefore E_1=\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a^2}{b}\sqrt{\frac{r}{\pi x-2r\log b-p\log(a^2-b^2)}}\right]$$

Hence, at a long distance from the edge, the charge on this face will be sensibly equal to

$$\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a^2}{b}\sqrt{\frac{r}{\pi x}}\right]$$

The charge on the under face of the top plate is obtained by putting $t_2=1$, $t_1=t$, where t is nearly equal to a. Then

$$E_2=\frac{V}{4\pi^2}\log\left[\frac{2a}{t-a}\cdot\frac{1-a}{1+a}\right]$$

But, for t nearly equal to, but greater than a

$$x=-\frac{2r}{\pi}\log\frac{a}{b}-\frac{p}{\pi}\log\frac{2a(t-a)}{a^2-b^2}+\frac{ra^2(a^2-b^2)}{b^2\pi}$$

$$\therefore E_2=+\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\left(\frac{1-a}{1+a}\cdot\frac{4a^2}{a^2-b^2}\right)-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

The charge on the top face of the next to the top plate is obtained by putting $t_1=b$, $t_2=t$ nearly $=a$. Then

$$E_3=\frac{V}{4\pi^2}\log\left(\frac{a-t}{2a}\cdot\frac{a+b}{a-b}\right).$$

But for t nearly equal to a

$$x=-\frac{2r}{\pi}\log\frac{a}{b}-\frac{p}{\pi}\log\frac{2a(a-t)}{(a^2-b^2)}+\frac{a^2r(a^2-b^2)}{b^2\pi}$$

$$\therefore E_3=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\frac{4a^2}{(a+b)^2}-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

The charge on the under side of the next to the top plane is obtained by putting $t_1=t_0$, a very small quantity, and $t_2=b$. Then

$$E_4=\frac{V}{4\pi^2}\log\left[\frac{a-b}{a+b}\cdot\frac{a+t_0}{a-t_0}\right]$$

$$=\frac{V}{4\pi^2}\left[\log\frac{a-b}{a+b}+\frac{2t_0}{a}\right]$$

But, for very small values of t

$$x=-\frac{2r}{\pi}\log\frac{t_0}{b}-\frac{p}{\pi}\log\frac{a^2}{a^2-b^2}-\frac{ra}{\pi}$$

$$\therefore t_0=b\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}$$

$$\therefore E_4=\frac{V}{4\pi^2}\left[\log\frac{a-b}{a+b}+\frac{2b}{a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}\right]$$

$$=-\frac{V}{4\pi^2}\left[\log\frac{a+b}{a-b}-\frac{2b}{a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}\right] \quad (15)$$

From these expressions we find the total charge on the inner cylinder between its end and a long distance, x, from the end is

$$E=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\frac{4a^2}{a^2-b^2}-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

These equations are sufficient to enable us to determine the edge effect and the variations of the density with the distance from the edge, provided this distance is not too small. However, in order to study the variation of density on the upper face of the next to the top plane it is advisable to obtain a general expression for the density. It is

$$\sigma_3 = -\frac{1}{4\pi}\frac{\partial\phi}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= -\frac{V}{4\pi p}\left[\frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{t}{(1-t^2)(t^2-b^2)}\right]$$

For $t=a$, the density is uniform and equal to

$$\sigma_0 = -\frac{V}{4\pi p}$$

Hence,

$$\frac{\Delta\sigma}{\sigma_0} = \frac{\sigma_3-\sigma_0}{\sigma_0} = \frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{t}{(1-t^2)(t^2-b^2)} - 1$$

If the density is sensibly equal to σ_0, this expression must be sensibly zero. If $t^2 = a^2(1-\epsilon)$, we find

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = \frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{a(1-\epsilon)^{\frac{1}{2}}}{(1-a^2+a^2\epsilon)(a^2-b^2-a^2\epsilon)} - 1$$

$$= \frac{(1-a^2)(a^2-b^2)(1-\epsilon)^{\frac{1}{2}}}{(1-a^2)(a^2-b^2)-a^2\epsilon\,[1-a^2-(a^2-b^2)]-a^4\epsilon^2} - 1$$

$$= (1-\epsilon)^{\frac{1}{2}}\left\{1+a^2\epsilon\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)+\ldots\right\} - 1$$

$$= \epsilon\left\{a^2\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)-\frac{1}{2}\right\}$$

Hence, in order that

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = +10^{-n}$$

we must have

$$t^2 = a^2\left[1 - \frac{10^{-\eta}}{a^2\left(\frac{1}{a^2-b^2} - \frac{1}{1-a^2}\right)-0.5}\right] \tag{16}$$

Knowing a^2 and b^2, and assuming a suitable value for η, we can find t^2, and from this can find x.

Case 2. *A condenser consisting of a thick semi-infinite plate midway between, and parallel to two infinitely thin semi-infinite parallel plates; the edges of all the plates lie in the same plane normal to the system of plates.*—The diagram in the z plane is as shown in Fig. 20. Assume

$t = +\infty$ at A	$t = -b$ at E
$= +1$ at B	$= -a$ at F
$= +a$ at C	$= -1$ at G
$= +b$ at D	$= -\infty$ at H

The transformation is

$$\frac{dz}{dt} = C\frac{t^2-1}{t^2-a^2}\sqrt{t^2-b^2}$$

$$\text{or, } \frac{dz}{dt} = C\left[t^2 - (1+b^2-a^2) - \frac{(1-a^2)(a^2-b^2)}{2a}\left\{\frac{1}{t-a} - \frac{1}{t+a}\right\}\right]\left(\frac{1}{\sqrt{t^2-b^2}}\right)$$

$$\therefore z = C\left[\begin{array}{l} \frac{t}{2}\sqrt{t^2-b^2} - \left(1+\frac{b^2}{2}-a^2\right)\log\left(t+\sqrt{t^2-b^2}\right) \\ -\frac{(1-a^2)}{2a}\sqrt{a^2-b^2}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2-at}{(t+a)\sqrt{a^2-b^2}}\right. \\ \qquad \left. -\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2+at}{(t-a)\sqrt{a^2-b^2}}\right\} \end{array}\right] + C_1 + iD_1$$

C_1 and D_1 being constants of integration.

If $z = 0$ at the point $t = 0$, we have

$$0 = C\left[-\left(1+\frac{b^2}{2}-a^2\right)\log b - i\left\{\left(1+\frac{b^2}{2}-a^2\right)\frac{\pi}{2} + \frac{\pi(1-a^2)\sqrt{a^2-b^2}}{2a}\right\}\right] + C_1 + iD_1$$

$$\therefore C_1 = C\left(1+\frac{b^2}{2}-a^2\right)\log b$$

$$D_1 = C\left[\left(1+\frac{b^2}{2}-a^2\right)\frac{\pi}{2}+\frac{\pi(1-a^2)\sqrt{a^2-b^2}}{2a}\right]$$

$$\therefore z = C\left[\begin{array}{l} \frac{t}{2}\sqrt{t^2-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\frac{t+\sqrt{t^2-b^2}}{b} \\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2-at}{(t+a)\sqrt{a^2-b^2}}\right. \\ \qquad \left.-\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2+at}{(t-a)\sqrt{a^2-b^2}}\right\} \\ +\frac{i\pi}{2}\left\{\left(1+\frac{b^2}{2}-a^2\right)+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\} \end{array}\right]$$

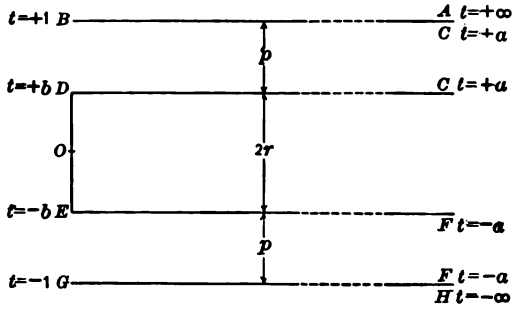

Fig. 20.

For $t=b$, we have $z=ir$

$$\therefore ir = C\left[-\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{-b(a+b)}{(a+b)\sqrt{a^2-b^2}}-\log\frac{b(a-b)}{-(a-b)\sqrt{a^2-b^2}}\right\}\right.$$
$$\left.+\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\}\right]$$

$$= C\left[1+\frac{b^2}{2}-a^2-\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]\frac{\pi i}{2}$$

$$\therefore r = C\left[1+\frac{b^2}{2}-a^2-\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]\frac{\pi}{2} \qquad (17)$$

For $t=1$, $z=i(b+r)$

$$\therefore i(p+r)=C\left[\begin{array}{l}\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\left(\frac{1+\sqrt{1-b^2}}{b}\right)\\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2-a}{(1+a)\sqrt{a^2-b^2}}\right.\\ \qquad\left.-\log\frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2+a}{(1-a)\sqrt{a^2-b^2}}\right\}\\ +\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\}\end{array}\right]$$

$$=C\left[\begin{array}{l}\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\left(\frac{1+\sqrt{1-b^2}}{b}\right)\\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1+a)\sqrt{a^2-b^2}}\right.\\ \qquad\left.-\log\frac{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1-a)\sqrt{a^2-b^2}}\right\}\\ +\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2\right\}\end{array}\right]$$

$$\therefore 0=C\left[\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\frac{1+\sqrt{1-b^2}}{b}\right.$$

$$\left.-\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\log\left(\frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}}\cdot\frac{(1-a)}{(1+a)}\right)\right] \quad (18)$$

and
$$C\left[1+\frac{b}{2}-a^2\right]\frac{\pi}{2}=p+r \quad (19)$$

As t decreases through the value a, z must decrease by ip

$$\therefore C\left[\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\right]\pi=p \quad (20)$$

Substituting this value of C in (17), we obtain the relation (19) which becomes

$$\left(1+\frac{b^2}{2}-a^2\right)=\frac{p+r}{p}\left[\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]$$

Solving this equation for b^2 we find

$$b^2=\frac{2(1-a^2)}{p^2a^2}\Big\{\sqrt{[p^2+(1-a^2)(2pr+r^2)]^2+a^4p^2(2pr+r^2)}-[p^2+(1-a^2)(2pr+r^2)]\Big\} \quad (21)$$

This and (18) serve to determine a and b when p and r are known. Substituting the value of C in the expression for z and rearranging terms, we find

$$z=\frac{p}{\pi}\cdot\frac{a\,t\sqrt{t^2-b^2}}{(1-a^2)\sqrt{a^2-b^2}}-\frac{2(p+r)}{\pi}\log\frac{t+\sqrt{t^2-b^2}}{b}$$
$$-\frac{p}{\pi}\log\left(\frac{a\,t+b^2-\sqrt{a^2-b^2}\sqrt{t^2-b^2}}{a\,t-b^2+\sqrt{a^2-b^2}\sqrt{t^2-b^2}}\cdot\frac{t-a}{t+a}\right)+i(p+r)$$

The diagram in the w plane is as before. Hence, the charge per unit length of the strip between the values $t=t_1$, and $t=t_2$; $t_2>t_1$ is

$$E=\frac{V}{4\pi^2}\log\left[\frac{t_2-a}{t_2+a}\cdot\frac{t_1+a}{t_1-a}\right]$$

The total charge on unit length of the middle plate between O and C is

$$E_t=\frac{V}{4\pi^2}\log\left[\frac{t-a}{2a}\cdot\frac{a}{-a}\right]=\frac{V}{4\pi^2}\log\frac{a-t}{2a}$$

where t is less than but nearly equal to a. But, for t nearly equal to a, we find

$$x=\frac{pa^2}{\pi(1-a^2)}-\frac{2(p+r)}{\pi}\log\frac{a+\sqrt{a^2-b^2}}{b}$$
$$-\frac{p}{\pi}\log\frac{b^2}{2a(a^2-b^2)}-\frac{p}{\pi}\log(a-t)$$

$$\therefore E_t=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\left\{2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}+\frac{2r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b}-\frac{a^2}{1-a^2}\right\}\right]$$

Hence, the effective length of the thick plate is greater than its actual length by an amount

$$\delta l_t=\frac{p}{\pi}\left[2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}+\frac{2r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b}-\frac{a^2}{1-a^2}\right] \quad (22)$$

31. EFFECT OF VARYING THE DISTANCE BETWEEN THE CLOSED ENDS OF CONCENTRIC CYLINDERS.

Lord Rayleigh[3] has shown that the force of attraction between two semi-infinite cylinders placed as shown in Fig. 21, and maintained at unit difference of potential is approximately

$$F=\Sigma\frac{(a+b)^2J_0^2[\frac{1}{2}k(a+b)]}{8b^3\sinh^2kl.\,J_0'^2(kb)}$$

the summation being with reference to k, which is given by the roots of the equation

$$J_0(kb)=0$$

Hence the values of kb are 2.404, 5.520, 8.657, etc. The upper limit for the force is obtained by putting $a=b$ in the above expression, when it becomes

$$\frac{1}{2}\Sigma[\sinh kl]^{-2}$$

But, the force is

$$F=\frac{dC}{dl}$$

the potential being constant and equal to unity. Hence,

$$\frac{dC}{dl}<\frac{1}{2}\sinh^{-2}k_1l \qquad \text{(approximately)}$$

where $k_1=\frac{2.404}{b}$. If $b=7.25$,

$l=12$, then $kl=3.98$

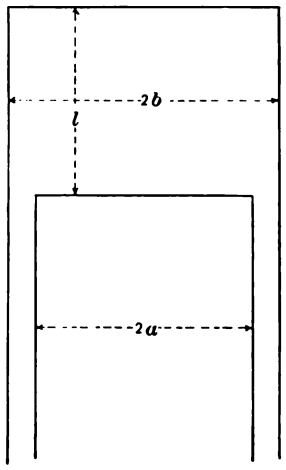

Fig. 21.

$$\therefore\frac{dC}{dl}<0.0007$$

Hence, even though l should vary by 0.1 mm the capacity would be changed by but 7×10^{-6}cm, that is by but one ten-millionth part of the capacity of one section of the condenser.

32. CORRECTION FOR THE GAP BETWEEN THE GUARD CYLINDER AND THE MIDDLE CYLINDER.

Here we must take into account the fact that the guard cylinder and the middle cylinder are not continuations of one another, but there is an offset where they come together. Proceeding as before, we shall now solve the problem for a condenser of planes, and shall treat the plates as though they were of infinite thickness. The diagram in the z plane is as shown in Fig. 22.

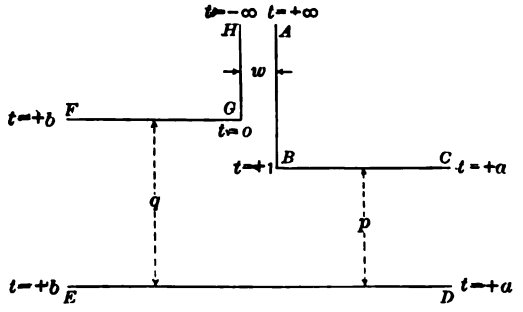

Fig. 22.

Assume $t = -\infty$ at $y = \infty$ on the HG face,
$t = 0$ at G,
$t = +b$ at $x = -\infty$ on the GF or the DE plane,
$t = +a$ at $x = +\infty$ on the BC or the DE plane,
$t = +1$ at B,
$t = +\infty$ at $y = \infty$ on the AB face.

The transformation is given by

$$\frac{dz}{dt} = C\frac{\sqrt{t}\,(t-1)}{(t-a)(t-b)}$$

$$\text{or, } \frac{dz}{dt} = iC\left[-1 - \frac{1}{a-b}\left\{\frac{b(1-b)}{t-b} - \frac{a(1-a)}{t-a}\right\}\right]$$

$$\therefore z = -iC\sin^{-1}(2t-1)$$

$$-\frac{iC}{a-b}\left\{\begin{array}{l}\sqrt{a(1-a)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{a(1-a)}+t(1-2a)+a}{2(t-a)\sqrt{a(1-a)}} \\ -\sqrt{b(1-b)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{b(1-b)}+t(1-2b)+b}{2(t-b)\sqrt{b(1-b)}}\end{array}\right\} + C_1$$

If we take B as origin, we have

$$0 = -iC\left[\frac{\pi}{2}+\frac{1}{a-b}\left\{\sqrt{a(1-a)}\log\frac{1}{2\sqrt{a(1-a)}} - \sqrt{b(1-b)}\log\frac{1}{2\sqrt{b(1-b)}}\right\}\right]+C_1$$

$$\therefore z = -iC\left\{\sin^{-1}(2t-1)-\frac{\pi}{2}\right\} - \frac{iC}{a-b}\left\{\begin{array}{l}\sqrt{a(1-a)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{a(1-a)}+t(1-2a)+a}{t-a} \\ -\sqrt{b(1-b)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{b(1-b)}+t(1-2b)+b}{t-b}\end{array}\right\}$$

As t decreases through the value a, z increases by $-ip$

$$\therefore -iC = \frac{p}{\pi}\frac{(a-b)}{\sqrt{a(1-a)}}$$

Similarly, as t decreases through the value b, z increases by $+iq$

$$\therefore -iC = \frac{q}{\pi}\frac{(a-b)}{\sqrt{b(1-b)}}$$

At G, $t=0$, $z=-w+i(q-p)$

$$\therefore -w+i(q-p) = i\pi C - \frac{\pi C}{a-b}\left\{\sqrt{a(1-a)}-\sqrt{b(1-b)}\right\}$$

$$\therefore -iC = \frac{w}{\pi} = \frac{p(a-b)}{\pi\sqrt{a(1-a)}} = \frac{q(a-b)}{\pi\sqrt{b(1-b)}}$$

Solving these equations for a and b, we find

$$a = \frac{1}{2}\left[\frac{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}+[q^2-p^2+w^2]}{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}}\right]$$

$$= \frac{1}{2}\left[1+\frac{q^2-p^2+w^2}{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}}\right] = \frac{1}{2}\left[1+\frac{q^2-p^2+w^2}{\sqrt{[q^2+p^2+w^2]^2-4q^2p^2}}\right]$$

$$b = \frac{1}{2}\left[\frac{\sqrt{[q^2-p^2-w^2]^2+4w^2q^2}+[q^2-p^2-w^2]}{\sqrt{[q^2-p^2-w^2]^2+4w^2q^2}}\right]$$

$$= \frac{1}{2}\left[1 + \frac{q^2 - p^2 - w^2}{\sqrt{[q^2 - p^2 - w^2]^2 + 4w^2q^2}}\right] = \frac{1}{2}\left[1 + \frac{q^2 - p^2 - w^2}{\sqrt{[q^2 + p^2 + w^2]^2 - 4q^2p^2}}\right]$$

Substituting the values of iC in the expression for z we find

$$z = \left[\begin{array}{l} \frac{w}{\pi}\left\{\sin^{-1}(2t-1) - \frac{\pi}{2}\right\} \\ + \frac{p}{\pi}\log\frac{2\sqrt{t(1-t)}\sqrt{a(1-a)} + t(1-2a) + a}{t-a} \\ - \frac{q}{\pi}\log\frac{2\sqrt{t(1-t)}\sqrt{b(1-b)} + t(1-2b) + b}{t-b} \end{array}\right]$$

The transformation from the w plane to the t plane is given by

$$\frac{dw}{dt} = \frac{B}{\sqrt{(t-a)(t-b)}}$$

$$\therefore w = \phi + i\psi = \frac{V}{\pi}\left[\log\frac{t-b}{t-a} + i\pi\right]$$

Now the density at any point is

$$\sigma = -\frac{1}{4\pi}\frac{d\psi}{dv} = -\frac{1}{4\pi}\frac{d\phi}{ds}$$

$$= -\frac{1}{4\pi}\frac{d\phi}{dt}\cdot\frac{dt}{ds}$$

$\frac{dt}{ds}$ is positive for all these surfaces.

Hence, the charge between t_1 and t_2, $t_2 > t_1$, is

$$E = \frac{V}{4\pi^2}\log\left(\frac{t_1-b}{t_1-a}\cdot\frac{t_2-a}{t_2-b}\right)$$

The charge on AB is found by putting $t_1 = 1$, $t_2 = t$, a very large quantity, hence

$$E_1' = \frac{V}{4\pi^2}\left[\log\left(\frac{1-b}{1-a}\right) - \frac{a-b}{t}\right]$$

in the limit this becomes

$$E_1' = \frac{V}{4\pi^2}\log\frac{1-b}{1-a}$$

The charge on BC is found by putting $t_2 = 1$, $t_1 = t$, nearly equal to a, hence

$$E_1'' = \frac{V}{4\pi^2} \log\left[\frac{1-a}{1-b} \cdot \frac{a-b}{t-a}\right]$$

But, for t nearly equal to a, we have

$$z = \left[\begin{array}{l} \frac{w}{\pi}\left\{\sin^{-1}(2a-1) - \frac{\pi}{2}\right\} + \frac{p}{\pi}\log 4a(1-a) \\ \quad - \frac{q}{\pi}\log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} + \frac{p}{\pi}\log\frac{1}{t-a} \end{array}\right]$$

$$\therefore \log\frac{1}{t-a} = \frac{\pi}{p}\left[\begin{array}{l} x + \frac{w}{\pi}\left\{\frac{\pi}{2} - \sin^{-1}(2a-1)\right\} - \frac{p}{\pi}\log 4a(1-a) \\ + \frac{q}{\pi}\log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \end{array}\right]$$

$$\therefore E_1'' = \frac{V}{4\pi p}\left[\begin{array}{l} x + \frac{w}{\pi}\left\{\frac{\pi}{2} - \sin^{-1}(2a-1) - \frac{p}{\pi}\log\frac{4a(1-b)}{a-b}\right. \\ + \frac{q}{\pi}\log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \end{array}\right]$$

Hence, the charge from A to a point on BC far from B is

$$E_1 = \frac{V}{4\pi p}\left[x + \delta l_1\right]$$

where,

$$\delta l_1 = \left[\begin{array}{l} \frac{w}{\pi}\left\{\frac{\pi}{2} - \sin^{-1}(2a-1)\right\} - \frac{p}{\pi}\log\frac{4a(1-a)}{a-b} \\ + \frac{q}{\pi}\log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \end{array}\right]$$

$$\text{or, } \delta l_1 = \left[\begin{array}{l} \frac{w}{\pi}\left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{q^2 - p^2 + w^2}{\sqrt{[q^2+p^2+w^2]^2 - 4p^2q^2}}\right)\right\} \\ - \frac{p}{\pi}\log\frac{4p^2}{\sqrt{[q^2+p^2+w^2]^2 - 4p^2q^2}} \\ + \frac{q}{\pi}\log\frac{(p+q)^2 + w^2}{\sqrt{[p^2+q^2+w^2]^2 - 4p^2q^2}} \end{array}\right] \qquad (23)$$

Similarly, the charge on HG is found by putting $t_1 = -\tau$, τ being very large, $t_2 = 0$

then $$E_2' = \frac{V}{4\pi^2}\left[\log\frac{a}{b} - \frac{a-b}{r}\right]$$

In the limit this becomes

$$E_2' = \frac{V}{4\pi^2}\log\frac{a}{b}$$

The charge between G and a point near F is found by putting $t_1 = 0$, $t_2 = t$, t nearly $= b$, then

$$E_2'' = \frac{V}{4\pi^2}\log\frac{b}{a}\cdot\frac{a-b}{b-t}$$

But, for t nearly equal to b, we have

$$x = \begin{bmatrix} \frac{w}{\pi}\left\{\sin^{-1}(2b-1) - \frac{\pi}{2}\right\} \\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ -\frac{q}{\pi}\log 4b(1-b) - \frac{q}{\pi}\log\frac{1}{b-t} \end{bmatrix}$$

$$\therefore \log\frac{1}{b-t} = \frac{\pi}{q}\begin{bmatrix} -x + \frac{w}{\pi}\left\{\sin^{-1}(2b-1) - \frac{\pi}{2}\right\} \\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ -\frac{q}{\pi}\log 4b(1-b) \end{bmatrix}$$

If X = distance from G to the point $(-x)$, $-x = X + w$

$$\therefore E_2'' = \frac{V}{4\pi q}\begin{bmatrix} X + \frac{w}{\pi}\left\{\frac{\pi}{2} + \sin^{-1}(2b-1)\right\} \\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ -\frac{q}{\pi}\log\frac{4a(1-b)}{a-b} \end{bmatrix}$$

Hence, charge from H to a point on FG at a great distance, X, from G is

$$E_2 = \frac{V}{4\pi q}[X + \delta l_2]$$

where

$$\delta l_2=\left[\begin{array}{l}\frac{w}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}(2b-1)\right\}\\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+b(1-2a)+a}{a-b}-\frac{q}{\pi}\log\frac{4b(1-b)}{a-b}\end{array}\right]$$

$$=\left[\begin{array}{l}\frac{w}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left(\frac{q^2-p^2-w^2}{\sqrt{[q^2+p^2+w^2]^2-4p^2q^2}}\right)\right\}\\ +\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}}\\ -\frac{q}{\pi}\log\frac{4q^2}{\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}}\end{array}\right] \qquad (24)$$

If we put $A\equiv\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}$ and rearrange the terms, equations (23) and (24) can be put in the more simple form

$$\delta l=\left[\begin{array}{l}\frac{w}{2}-\frac{w}{\pi}\sin^{-1}\frac{w^2+q^2-p^2}{A}+\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{4p^2}\\ +\frac{q-p}{\pi}\log\frac{(p+q)^2+w^2}{A}\end{array}\right] \qquad (25)$$

with the understanding that δl is the correction for the portion of the plate corresponding to p; nothing is implied as to the relative magnitudes of p and q.

33. NUMERICAL VALUES OF THE CORRECTION TERMS AND ERRORS.

1. **Cylinders without guards.**—Let us now apply these results to the cases that occur in our experimental work. We shall first consider the case in which the plates are of infinitesimal thickness.

(*a*) *At what distance from the end does the distribution of the charge on the opposed surfaces become uniform?*—Referring to equation (16), we find that, if t^2 is equal to, or greater than, the value defined by the expression

$$t^2=a^2\left[1-\frac{10^{-5}}{a^2\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)-0.5}\right]$$

the density of the charge differs from its value at an infinite distance from the end by only 1 in 100,000.

Taking r as the radius of the inner cylinder = 6.25 cm and p as 1 cm we find

$$a^2 = 0.6926$$
$$b^2 = 0.4555$$
$$t^2 = a^2 (1 - 5.85 \times 10^{-5})$$
$$\therefore t^2 = 0.69256$$

But from equation (13) we know

$$x = -\frac{r}{\pi} \log \frac{t^2}{b^2} - \frac{p}{\pi} \log \frac{a^2 - t^2}{a^2 - b^2} + \frac{r}{\pi} \frac{a^2(t^2 - b^2)}{b^2}$$

Hence, the distance beyond which the density differs from its value at infinity by an amount which is less than 1 in 100,000 is

$$x = 2.52 \text{ cm.}$$

Hence, the distribution on the opposing surfaces at the central section of a pair of cylinders 12 cm long can not differ by an appreciable amount from its value at the center of a pair of infinitely long cylinders.

(*b*) *Effect of charge on the interior of the inner cylinder.*—If the cylinders are uncapped, there will be a certain charge on the interior of the inner one, and the magnitude of this charge will increase as the length of the cylinder is increased. It is necessary now to form an estimate of this charge. Throughout the work the lower end of the inner cylinder was always electrically closed. Hence, as the cylinder is reduced in length, the charge on its inner wall will decrease, but the charge on the upper surface of the cap on the lower end will increase. That is, this cap tends to compensate for the removal of the further end of the cylinder, and, if the cylinder is very short, it will overcompensate. Hence, the increase in the total charge on the interior of the inner cylinder as its length is increased from 12 to 32 cm is less than the charge that lies upon a length of 20 cm of the interior, between 12 and 32 cm from the end of a semi-infinite cylindrical condenser. For a series of four thin plates we find, by a slight modification of equation (15) the charge on the interior and between x_1 and x_2 centimeters from the end to be

$$E=\frac{V}{4\pi^2}\cdot\frac{2b}{a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{a^2}{2}}\left\{e^{-\frac{\pi x_1}{2r}}-e^{-\frac{\pi x_2}{2r}}\right\}$$

The charge per unit length upon the opposed surfaces is

$$\sigma_0=\frac{V}{4\pi p}$$

$$\therefore\frac{E}{\sigma_0}=\frac{2bp}{\pi a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{a^2}{2}}\left\{e^{-\frac{\pi x_1}{2r}}-e^{-\frac{\pi x_2}{2r}}\right\}$$

If $p=1$, $r=6.25$; then $a^2=0.6926$, $b^2=0.4555$; if we also take $x_1=12$, $x_2=32$ we find

$$\frac{E}{\sigma_0}=0.0165 \text{ cm}$$

Hence, the variation of the charge upon the interior of the inner cylinder with the length of the cylinder, is probably inappreciable.

This conclusion is borne out by the results of experiments on the effect of capping the inner cylinder. These experiments showed that the increase in capacity, produced by capping the condenser by means of a cap fitting inside the inner cylinder and resting upon the stiffening flange near its upper end, when the condenser was 32 cm high, did not differ from that when it was 12 cm high by more than 0.0005 micro-microfarads.

(*c*) *Effect of the charge on the outside of the outer cylinder.*—In considering the system of four planes, we found that there is an appreciable charge on the outside of the outer planes even at great distances from the end. The same must be true for the cylinders. Hence, the shortening of the cylinders will tend to decrease the capacity of the remaining portion. This decrease will depend upon the magnitude of the charge originally upon the portion which has been removed, and also upon the distribution of the charge; the further the electrical center of this charge is from the end of the cylinder, the less will the removal of this portion of the outer cylinder affect the capacity of the remaining portion. Being unable to give at present a fair estimate of the actual amount by which the shortening of the cylinder by a given amount will decrease the capacity of the remaining portions, we shall limit ourselves to the calcula-

tion of the charge on sections of the outside of the outer plane at

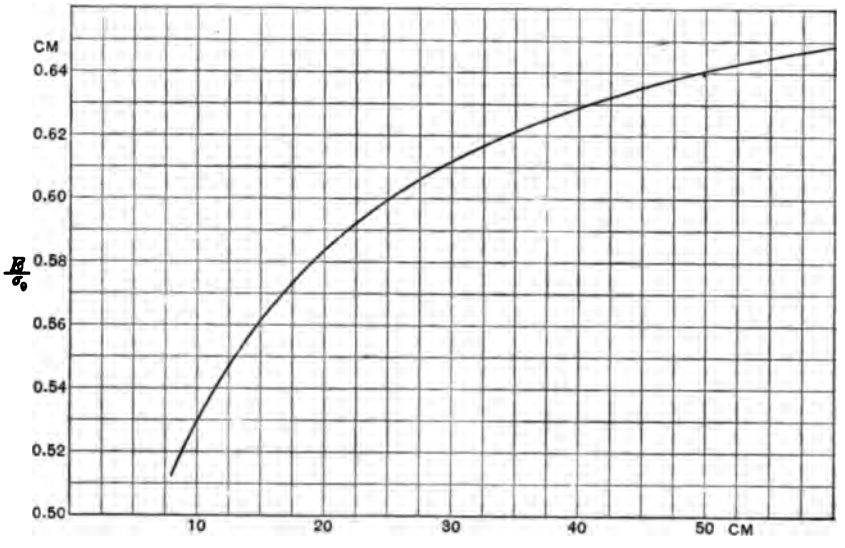

Fig. 23.—*Curve Showing Distribution of Charge on Outside of Outer Cylinder.*

different distances from the end. For this purpose we shall make use of equations (13) and (14), which, for convenience, we may rewrite as follows:

$$x=\frac{r}{\pi}\left[\frac{a^2(t^2-b^2)}{b^2}-\log\frac{t^2}{b^2}\right]-\frac{p}{\pi}\log\frac{t^2-a^2}{a^2-b^2}$$

$$\frac{E}{\sigma_0}=\frac{p}{\pi}\left[\log\frac{1+a}{1-a}-\frac{2a}{t_1}-\frac{2}{3}\left(\frac{a}{t_1}\right)^3\cdots\right]$$

where t is greater than a. E is the total charge on a strip of the outside of the outer plane, of unit length normal to the plane of the paper, and extending from the edge of the plane to the point where t has the large value t_1; σ_0 is the density on the under surface of this plane and at a long distance from the edge. Taking $p=1$, $r=6.25$, we find $a^2=0.6926$, $b^2=0.4555$; and from these values we obtain the curve connecting the ratio $\frac{E}{\sigma_0}$ with x. This curve is given in Fig. 23, and from it we find that the charge on the section extending from $x=12$ to $x=32$ is equivalent to the charge on a length of 0.72 mm of the opposed surfaces; this is equal to 36 parts in 10,000

of the whole charge on this section. Similarly the charge between $x=32$ and $x=52$ is equal to the charge on a length of 0.26 mm of the opposed surfaces; this equals 13 parts in 10,000 of the whole charge on this section. Hence, it is very possible that the charge on the end of the inner cylinder may be increased by 10 parts in 10,000 of the charge on a 20 cm section of the condenser when the height of the cylinder is increased from 12 cm to 32 cm.

(*d*) *Correction for different diameters.*—There is a further correction due to the fact that the ends of the various sections of the cylinders have slightly different diameters. This will cause the charges on the ends of the various sections to differ. To get an idea as to the magnitude of this variation when the outer cylinder is not capped, it is necessary to study the distribution of charge on the end of a semi-infinite thick plate midway between two equal semi-infinite plates of finite thickness. This problem, however, is more difficult than the one for which the outer planes are of infinitesimal thickness, and its result must lie between that given by the latter and that obtained from the problem of a semi-infinite thick plate midway between two infinite plates. This last problem has been solved,[9] and applies approximately to the case in which the outer cylinder extends some distance beyond the ends of the inner one, and both are capped.

The expression for the excess over its true length of the effective length of a thick plate between two semi-infinite plates of infinitesimal thickness is given by equation (22). It is

$$\delta l_t=\frac{p}{\pi}\left[\frac{2r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b}+2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}-\frac{a^2}{1-a^2}\right]$$

where $2r$ is the thickness of the plate, $2(p+r)$ is the distance between the thin plates, and a and b are functions of the ratio of r to p.

With the same notation, the expression given by J. J. Thomson for this quantity, when the outermost plates are of infinite length, is

$$\delta l_T=\frac{p}{\pi}\left[\frac{r}{p}\log\frac{2p+r}{r}+2\log\frac{2p+r}{p}\right]$$

If we take $p=1$, $r=6.25$, the quantities a and b have the values $a^2=0.80340$, $b^2=0.68783$; while if $p=1$, $r=6.25$ (1.001), we find

[9] J. J. Thomson, Recent Researches, § 237.

$a^2=0.80355$, $b^2=0.68804$. From these we find the values in the following table:

p	r	δl_1	δl_r
1	6.25	1.110	1.896
1	6.25 (1.001)	1.110	1.896

Hence, the quantity in the brackets varies but slowly as the ratio of r to p is changed. But the charge per unit length of the plate is inversely proportional to p, hence the additional charge added depends only upon the quantity in the brackets, and so is almost independent of p. Changing the ratio of p to r by so much as 1 in 1,000 will, in our case, change the effective length of the condenser by less than 10 μ, which is only 5 parts in 100,000 of the length of one section.

This concludes the mathematical discussion of the problem of cylinders without guard cylinders; and it is evident that if both cylinders are not capped at both ends, there is a considerable error introduced by the variation in the charges on the ends of the inner cylinder, produced by the change in the distribution of the charge on the outside of the outer cylinder as the length of the condenser is varied. Even were both cylinders capped at both ends it would still be necessary to determine experimentally the effect, if any, of the varying height of the cylinder upon the capacity of the lead wire and commutator.

In this work, the lower end of the inner cylinder was always closed, but the lower end of the outer one was never closed; it was, however, only 17 cm above a large (50 x 70 cm) earthed sheet of metal, and the heavy brass ring carrying the feet of the condenser was separated from the lower end of the outer cylinder by but 2 cm of ebonite (the base of the instrument) and was also earthed. We think this adjustment very completely shielded the lower end of the inner cylinder from variations in charge produced by a change in the height of the outer cylinder. For heights above 24 cm this was tested experimentally as follows:

Sections No. 1 being in place, outer cylinder No. 4 was placed on

outer cylinder No. 1, and was closed with a brass plate. The speed being adjusted and held constant by an auxiliary observer, the bridge was exactly balanced. Then, leaving everything else as before, another outer section was placed on top of the brass plate closing No. 4. On repeated trial this was found to produce an inappreciable increase in capacity, certainly not greater than 1 in 200,000 of the maximum capacity of the condenser.

Unfortunately, similar tests can not be made for the upper end of the cylinder. We have, however, made a few runs in which the upper ends of both cylinders were capped. This should completely eliminate the effect of varying the length of the outer cylinder. This being eliminated the remaining errors which we have estimated that we still have to contend with are

Lateral displacement . . .	$\frac{\Delta_1 C}{C_0} < + \ 0.2 \times 10^{-6}$
Inclination of the axis . . .	$\frac{\Delta_2 C}{C_0} < + \ 1.0 \times 10^{-6}$
Errors in figuring (estimated)	$\frac{\Delta_3 C}{C_0} < + 20.0 \times 10^{-6}$
Offsets	$\frac{\Delta C}{C_0} < + \ 1.0 \times 10^{-6}$
Variation in distances between the surfaces at the ends of the sections . . .	$\frac{\Delta C}{C_0} < \ 50.0 \times 10^{-6}$
Variation in position of cap .	$\frac{\Delta C}{C_0} < \ 1.0 \times 10^{-6}$
Total	$\frac{\Delta C}{C_0} < + 70 \times 10^{-6}$

Hence, the actual capacity of the cylinders without guard cylinders, but closed at both ends, may exceed the calculated capacity by about 7 in 100,000. To these errors is to be added the uncertainty introduced by variations in the capacity of the commutator.

2. **Cylinders with guards.**—In this part of the work we are not concerned with the charge on the outside of the outer cylinder or upon the ends of the inner cylinder; and, each guard cylinder being electrically closed, there can be no charge on the inside of the inner cylinder. We have also seen (p. 518) that the irregular distribution of charge on the outer surface of the inner cylinder does not

extend beyond 2.52 cm from the end (to an accuracy of 1 in 100,000 in the density), hence there is no correction to be applied for this irregularity. Also, as we do not have to stop to build up the condenser, we can pass rapidly from a determination of the capacity of the condenser, leads, and commutator to that of the leads and commutator only, and back again. Thus, variations in the capacity of the commutator can produce but a slight effect in the result of even a single determination. The other corrections considered under the discussion of the correction for cylinders without guards apply here, and in addition we must determine the correction for the width of the gap between the guard cylinder and the portion of which the capacity is measured.

In equation (25) we have the expression for the correction for a gap in one electrode of an infinite plate condenser, the distance between the plates being different on the two sides of the gap. Whence we find that the charge on the plate on one side of the gap is

$$E=\frac{V}{4\pi p}(x+\delta l)$$

where

$$\delta l=\frac{w}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]+\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{4p^2}$$
$$+\frac{q-p}{\pi}\log\frac{(p+q)^2+w^2}{A}$$

In this expression p is the distance between the plate under consideration and the plate ED (Fig. 22), q is the corresponding distance for the plate on the other side of the gap, w is the width of the gap, and $A\equiv\sqrt{(q^2+p^2+w^2)^2-4p^2q^2}$

If $q=p(1+\beta)$, where β is a small quantity, we may write

$$\delta l=\frac{w}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]+\frac{p\beta}{\pi}\left[1+\log\frac{(p+q)^2+w^2}{A}-\frac{\beta^2}{4}+\frac{w^2}{4p^2}\right]$$

These expressions are obtained on the hypothesis that the face ED is a continuous plane. If, however, there is an offset in the opposite plane, as shown in Figs. 24, 26, or 27, the problem is more

complicated. If HGF represents the portion corresponding to the guard cylinder, then in the cases with which we are concerned the step B′ is never in the plane AB. Now, since the gap correction depends almost entirely upon the rearrangement of the tubes of induction that would end upon the portion GG′ (Fig. 24) of the plate if the gap were absent, it is evident that in calculating the correction for the gap we must take as p, in our formulæ, the distance between BC and that plane which is cut by the prolongation of AB; q will then be $p+a_1$ where a_1 is the excess of the radius of the inner cylinder of the condenser proper over the radius of the guard cylinder. The correction thus obtained is given in terms of the capacity per unit of length of the condenser for which the distance between the plates is p. If A′B′ is on the guard cylinder side of AB, p will be the distance between the plates of the condenser the capacity of which we measure; if A′B′ is on the other side of AB, p will be less than this distance by an amount a_2 (where a_2 is the excess of the radius of the outer cylinder of the condenser proper over the radius of the outer cylinder of the guard section).

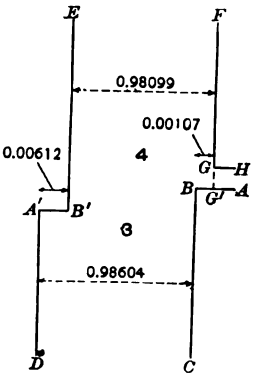

Fig. 24.

The calculated correction in this case must be multipled by $\left(1+\frac{a_2}{p}\right)$ in order to express it in terms of the capacity of unit of length of the section BC.

Now in the actual cases that occur in this work A′B′ is always displaced 0.06 cm from the plane of AB (0.06 cm is the width of the lower gap), except when the middle cylinder consists of one section only. Since the total correction due to the step A′B′ is small, and this displacement is small, no appreciable error can be introduced by assuming that the actual effect of A′B′ is the same as if there were no gap and the two steps were in the same plane. We shall make this assumption. For the upper gap the offsets were as shown in Fig. 24, $a_1=0.00107$ cm, $a_2=0.00612$ cm; distance between FG and EB′ $=0.98099$ cm; distance between BC and

A′D=0.98604 cm. Hence for the gap formula we must take

$$p=0.97992$$
$$q=0.98099$$

From these we obtain Table XII, in which w=width of gap; δl= increase of capacity produced by the gap, expressed in terms of the capacity of unit of length of a condenser for which the distance between the plates is 0.97992; $\delta l'$ is the product of δl by

$$\left(\frac{0.98371}{0.97992}\right)\frac{1}{40}$$

where 0.98371 is the mean of the mean distances between the plates for the two sections, and 40 is the sum of the lengths of the two sections; hence $\delta l'$ is the correction in terms of the capacity of the total condenser formed of section 2 and section 3; the fourth column contains the differences between the value of $\delta l'$ and the value of $\delta l'$ for $w=0$; the last column contains the product of the values in the preceding column by $\left(1+\frac{1}{4}\frac{p}{r}\right)$.

TABLE XII.

Variation of Capacity with Width of Guard Gap.

w	$\delta 1$	$\delta 1'$	$\delta 1'-\delta 1'_0$	$(\delta 1'-\delta 1'_0)\left(1+\frac{1}{4}\frac{p}{r}\right)$
0.00	0.00290	7.3×10^{-5}	0.0×10^{-5}	0.0×10^{-5}
.01	.00679	17.0 "	9.7 "	10.1 "
.02	.01150	28.8 "	21.5 "	21.4 "
.05	.02584	64.8 "	57.5 "	59.8 "
.10	.04941	124.0 "	116.7 "	121.3 "
.20	.09414	236.2 "	228.9 "	238.0 "
.30	.13615	341.7 "	334.4 "	347.6 "
.40	.17497	439.1 "	431.8 "	448.8 "
.50	.21072	528.8 "	521.5 "	542.1 "

The factor $\left(1+\frac{1}{4}\frac{p}{r}\right)$ is the correction introduced by J. J. Thomson[10]

[10] Phil. Trans., 181 A, p. 590; 1890.

on account of the curvature of the plates. The justification of this

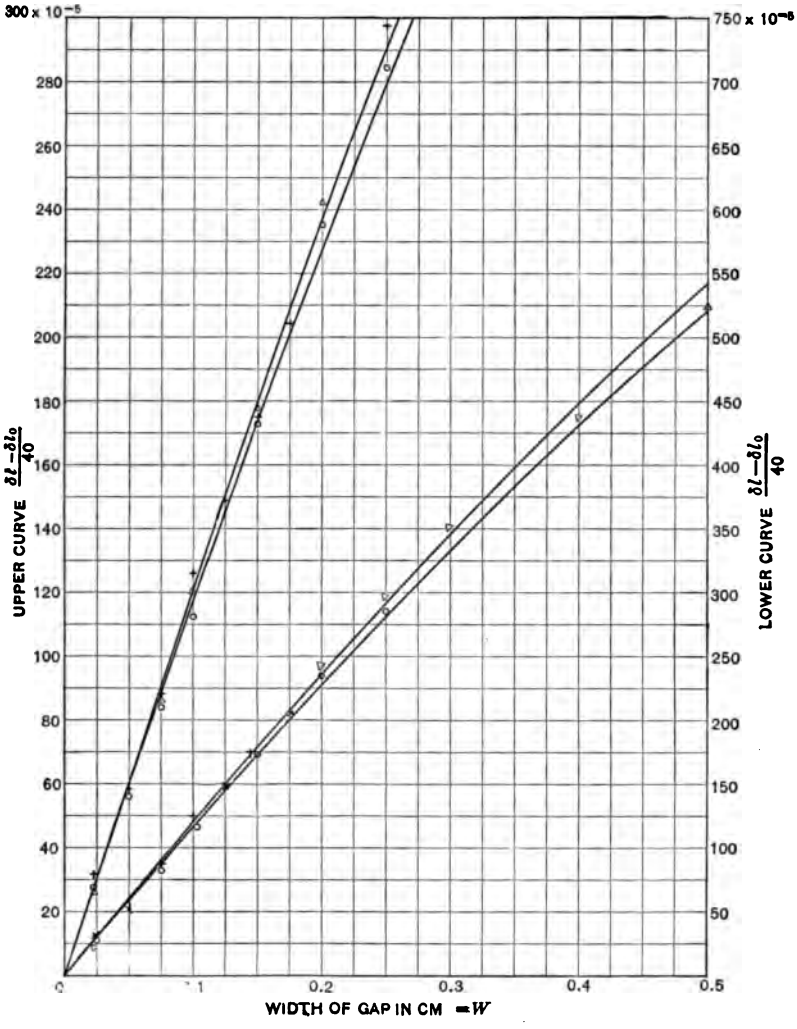

Fig. 25.—*Curve Showing Variation of Capacity with Width of Guard Gap.*

particular factor is not obvious. In Maxwell's Treatise, § 200, there is a similar factor of curvature deduced for a system of coaxial cylinders, *all of which* are at the same potential and placed end-on at a certain distance from a plane at a different potential; and its justifi-

cation is based explicitly upon the fact that the electric force in the direction of the radius of curvature is negligible except near the edges of the ends of the cylinders. None of these conditions are fulfilled in the case which we are considering, so this deduction can not be appealed to in justifying the factor.

The variation of the capacity with the width of the gap was measured for gaps of 0.25 mm to 5.0 mm, and the results of three independent series of such observations are plotted in Fig. 25, and the curves determined by the figures in the last two columns of Table XII are drawn on the same plot. From this it is evident that the experimental results lie between the two curves, and, on the whole, rather nearer the lower curve than the one which contains the $\left(1+\frac{1}{4}\frac{\rho}{r}\right)$ factor. The measurements are not of sufficient accuracy, however, to enable us to decide as to the correctness of the $\left(1+\frac{1}{4}\frac{\rho}{r}\right)$ factor. For gaps of 0.5 mm the difference in the correction as given by the two expressions is only 2×10^{-5} of the total capacity we are measuring, and so is negligible; in this work we shall (for such small gaps) assume that the simpler expression is correct.

Except in one instance, the width of the upper gap was 0.05 cm during the entire course of the work; hence, the total correction, in terms of centimeters of length of section No. 3, for the upper gap, offsets, etc., is as follows:

Correction for 0.05 cm gap, opposite face plane

$$=0.02584\times\frac{0.9809}{0.9799}=0.02587$$

Correction for offset in outer cylinder

$$=\left\{\begin{matrix}0.21\times 0.00612\\-0.32\times 0.00612\times 0.0017\end{matrix}\right\}=0.00128$$

Correction for elevation of upper end of inner section No. 3 by 0.06 cm above top of outer section No. 3

$$=0.06\times\frac{0.00612}{0.986}\times 0.981=0.00036$$

Total end correction . . . . . $=0.02751$

If the condenser is composed of section No. 2 and section No. 3 its length will be 40 cm and the mean distance between the charged surfaces will be 0.98371 cm; hence, for this case the above correc-

tion will amount to $\frac{0.02751}{40} \times \frac{0.9837}{0.9809} = 69.0$ parts in 100,000 of the total.

If the condenser is composed of section No. 3 only, the correction will be $\frac{0.02751}{20} = 137.6$ parts in 100,000 of the total, if the lower gap is 0.06 cm; if the gap is 0.05 cm the correction will be $\frac{0.2745}{20} = 137.2$ parts in 100,000 of the total.

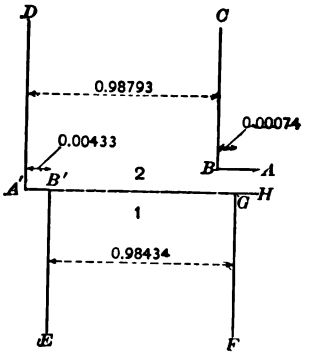

Fig. 26.

For the lower gap, also, two cases are to be considered, according as section No. 2 or section No. 3 adjoins section No. 1. Here the gap width is 0.05 cm in the former case and 0.06 cm in the latter.

With section No. 2 next to section No. 1, the dimensions are as shown in Fig. 26. From this data we find the correction for a gap of 0.06 cm to be equivalent to a length of 0.03028 cm of a condenser for which the plate distance is 0.98793. Since the mean distance for section No. 2 is 0.98651 we find the correction in terms of centimeters of length of section No. 2 to be as follows:

Correction for 0.06 cm gap, opposite face plane

$$= 0.03028 \times \frac{0.9865}{0.9879} = \quad 0.03024$$

Correction for offset in outer cylinder

$$\left.\begin{matrix} 0.21 \times 0.00433 \\ +0.32 \times 0.00433 \times 0.00074 \end{matrix}\right\} = \quad 0.00091$$

Total correction . . . . . = 0.03115

If the condenser is composed of both section No 2 and section No. 3 this correction is equal to $\frac{0.03115}{40} \times \frac{0.9837}{0.9865} = 77.6$ parts in 100,000 of the total.

With section No. 3 next to section No. 1, the dimensions are as shown in Fig. 27. From this data we find the correction for a gap of 0.05 cm to be equivalent to a length of 0.02572 cm of a condenser for which the plate distance is 0.98351 cm. Since the mean

distance for section No. 3 is 0.98091 cm, we find the correction in terms of centimeters of length of section No. 3 to be as follows:

Correction for 0.05 cm gap, opposite face plane

$$= 0.02572 \times \frac{0.9809}{0.9835} = \quad 0.02565$$

Correction for offset in outer cylinder

$$\left.\begin{matrix} -0.21 \times 0.00067 \\ -0.32 \times 0.00067 \times 0.00016 \end{matrix}\right\} = -0.00014$$

Total correction . . . . . = 0.02551

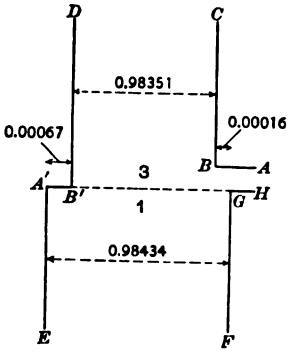

Fig. 27.

For this case the main condenser consists of section No. 3 only; hence, this correction amounts to 127.6 parts in 100,000 of the whole.

There are also offsets where section No. 2 and section No. 3 meet, but as shown on page 501 the effect of such small offsets occurring in the middle of a long condenser has the effect of increasing the effective length of the condenser by a negligible amount. Hence, when the guard cylinder condenser consists of section 2 and section 3, plus the guard cylinders, its electrostatic capacity in centimeters is the sum of the following quantities:

Sum of the two sections regarded as portions of infinite cylinders . . . . . = 137.004
Correction for upper gap and offset $= 137.0 \times 69.0 \times 10^{-5}$. = 0.095
Correction for lower gap and offset $= 137.0 \times 77.6 \times 10^{-5}$. = 0.106

Total capacity at 20°.0 C . . . . . = **137.205**

If the main condenser consists of section No. 3 only, its capacity is

Section No. 3 regarded as portion of an infinite cylinder . = 68.696
Correction for upper gap and offset $= 68.70 \times 137.2 \times 10^{-5}$ = 0.094
Correction for lower gap and offset $= 68.70 \times 127.6 \times 10^{-5}$ = 0.088

Total capacity at 20°.0 C . . . . . = **68.878**

VI. CORRECTIONS AND SOURCES OF ERROR—PLATE CONDENSER.

The corrections and sources of error that must be considered in the calculation of the electrostatic capacity of the plate condenser are as follows:

(1) Stability, scale error, error in setting, temperature changes.

(2) Inclination of the normal of the movable plate to the direction of motion.

(3) Lack of parallelism of the plates.

(4) Finite width of guard ring.

(5) Correction for the width of the gap between the plate and the guard ring.

(6) Displacement of the plate from the plane of the guard ring.

34. STABILITY, SCALE ERROR, ERROR IN SETTING, TEMPERATURE CHANGES.

By means of a silk thread passing through a hole in the top of the instrument, a 50 g weight can be lowered upon the collector plate of the instrument or lifted from it. It was found that the lowering of the weight upon the plate produced no appreciable change in the electromagnetic capacity. Also, by means of a Zeiss vertical comparator, the change in the position of the movable plate when a 100 g weight was placed upon it, was measured and found to be less than 0.1μ. Hence the stability of the plates is all that is required.

The position of the lower plate was determined by means of a silver scale attached to the vertical column carrying the lower plate, and read by a fixed microscope. This scale was graduated to 0.1 mm divisions, and was calibrated by the Division of Weights and Measures of this Bureau, and the corrections, so determined, have in every case been applied.

Every setting of the plate was so made that a millimeter division of the scale coincided with the zero setting of the microscope to within two microns. Then, five settings upon the line were made with the micrometer of the microscope, and the mean taken. The microscope was by Zeiss, and each division of the divided head corresponded to one micron. Except in rare instances the extreme range in any group of settings did not exceed 0.5 μ, and was usually less. The microscope mounting was very rigid. A load of 100 g

on the eye-piece end of the microscope changed the reading by only 1.2 μ; on the removal of the load the reading returned to its original value to within less than 0.1 μ. Hence, we think the settings are known to within 0.2 μ at least.

The method of procedure adopted in the work with the plate condenser is based upon the assumption that the distance between the plates, for a given scale reading, is the same throughout each entire set of observations. For an instrument composed of brass, steel, and ebonite this necessitates very constant temperature conditions; and at first these conditions were not fulfilled. This rendered valueless the earlier observations. For the later work the condenser was inclosed in a glass case in which, by electrical heating, the temperature could be raised to one or two degrees above the room temperature. By frequent measurement of the resistances of two groups each of three coils of copper wire, distributed about the condenser, it was possible to detect slight changes in the temperature, and, by suitable hand regulation of the heating current, to maintain the temperature constant to within one or two tenths of a degree. This procedure sufficiently eliminates this source of error.

35. INCLINATION OF THE NORMAL OF THE MOVABLE PLATE TO THE DIRECTION OF MOTION.

The only error introduced by the inclination of the normal to the movable plate to the direction of motion is that due to the fact that the measured motion of the column carrying the plate is greater than the change in the distance between the plates in the ratio of one to the cosine of the angle of inclination. By the method of adjustment described on page 454 it is easy to determine this angle and to reduce it sufficiently to make this error negligible. If this angle is not more than one-fourth of one degree the error introduced will be only 1 in 100,000.

36. LACK OF PARALLELISM OF THE PLATES.

As we saw on page 487 (equation 7) the capacity of a rectangular section of length l, normal to the axis of inclination, and of breadth b of a pair of slightly inclined plates is

$$C=\frac{bl}{4\pi d_1}\left(1+\frac{1}{3}\frac{\delta^2}{d_1^2}+\frac{1}{5}\frac{\delta^4}{d_1^4}+\cdots\right)$$

where d_1 is the mean distance between the plates, and δ is half of the difference between the greatest and the least distances between the plates in this section. We may write this expression in the form

$$C = \frac{bl}{4\pi d_1'}$$

where $$d_1' = d_1\left[1 - \frac{1}{3}\frac{\delta^2}{d_1^2} - \frac{4}{45}\frac{\delta^4}{d_1^4} - \quad . \quad . \quad .\right]$$

We shall call d_1' the effective distance between the plates.

If we now increase the distance between the plates until d_1 becomes d_2 the effective distance will become

$$d_2' = d_2\left[1 - \frac{1}{3}\frac{\delta^2}{d_2^2} - \frac{4}{45}\frac{\delta^4}{d_2^4} - \quad . \quad . \quad .\right]$$

$$\therefore d_2' - d_1' = d_2 - d_1 + \frac{d_2 - d_1}{3 d_2 d_1}\delta^2 + \frac{4}{45}\frac{d_2^3 - d_1^3}{d_2^3 d_1^3}\delta^4 + \quad . \quad . \quad .$$

$$= d_2 - d_1 + e$$

Where e is the quantity defined by the equation

$$e = \frac{(d_2 - d_1)\delta^2}{3 d_2 d_1}\left[1 + \frac{4}{15}\frac{d_2^2 + d_2 d_1 + d_1^2}{d_1^2 d_2^2}\delta^2 + \quad . \quad . \quad .\right] \qquad (26)$$

Hence, the effective distance that the plate has been moved is greater than the actual distance by the amount e.

By the method of procedure adopted in this work we measure d_1' and $d_2 - d_1$, and assume that $d_2' = d_1' + d_2 - d_1$. Hence the relative excess of the true value of the electrostatic capacity for the distance d_2 over its assumed value is

$$\frac{\Delta_1 C}{C_0} = -\frac{e}{d_2'} = -\frac{d_2 - d_1}{3 d_2^2 d_1}\delta^2, \text{ approximately.} \qquad (27)$$

It is evident, other things being equal, that this error is greater the smaller the initial distance, d_1; and for a given value of d_1 it is greater the smaller the value of d_2, provided d_2 is not too nearly equal to d_1. In Table XIII we give the values of $\frac{\Delta_1 C}{C_0}\cdot\frac{10^6}{\delta_\mu^2}$. To get $\frac{\Delta_1 C}{C_0}$ for any value of δ, therefore, we multiply the tabular value of $\frac{\Delta_1 C}{C_0}\frac{10^6}{\delta_\mu^2}$ by $\delta^2 10^{-6}$, expressing δ in microns. Thus if d_1 is 0.05 and $d_2 = 0.15$, then $\frac{\Delta_1 C}{C_0}$ will amount to 0.0001 if $\delta = 18.4\mu$, which corresponds to an inclination of but 38″. If δ were so much as 100μ the inclination would be but 3′ 26″, while the assumed capacity would be in error by nearly 3 parts in 1,000.

TABLE XIII.

Error Produced by Inclination of Plates.

$d_1=0.05$		$d_1=0.06$	
d_2	$\frac{\Delta_1 C}{C_0}\times\frac{10^6}{\delta^2_\mu}$	d_2	$\frac{\Delta_1 C}{C_0}\times\frac{10^6}{\delta^2_\mu}$
0.15	−0.296	0.16	−0.217
0.25	.213	0.26	.164
0.55	110	0.56	.088
1.05	.060	1.06	.049
2.05	.032	2.06	.026
2.55	.026	2.56	.021

The importance of this adjustment was not recognized until the conclusion of the work, and the lack of proper care in this respect explains one of the most annoying characteristics of all the work on the plates, viz, the values of **v** obtained for the shorter distances were almost invariably higher than the values found from the other condensers. If δ were 50μ, which might easily have been the case, the calculated electrostatic capacity for $d_2=0.15$ would be too high by 7.4 in 10,000 and the value found for **v** would be too high by 3.7 in 10,000, which is of the same order of magnitude as the variations actually observed.

Furthermore, while the upper plate was ground until it was very truly plane, the movable plate had a surface which was appreciably cylindrical. This error in figuring would of itself introduce an error of the kind which we have just considered.

37. CORRECTION FOR THE FINITE WIDTH OF THE GUARD RING.

J. J. Thomson[11] has shown that the charge on a strip of unit length normal to the paper and extending from the edge C (Fig. 28) of one of a pair of semi-infinite parallel plates to a point P at a distance x, is (if we consider the opposed surfaces only)

$$E=-\frac{\frac{1}{2}I'}{4\pi^2}\log(t+1)$$

[11] Recent Researches, § 235.

where the distance (x) from the edge is given by the expression

$$x=\frac{p}{2\pi}[t-\log\,(1+t)]$$

where t lies between -1 and 0, and p is the distance between the plates. Putting in the last expression

$$t=-1(1-10^{-n})$$

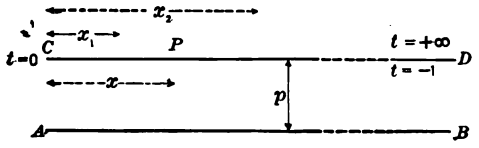

Fig. 28.

we find

$$\frac{2\pi x}{p}=2.302\;n+10^{-n}-1$$

Hence, if n is not less than 2, that is, if $\frac{2\pi x}{p}$ is not less than 3.614, the value of $\frac{2\pi x}{p}$ is given with an accuracy of at least 1 in 300 by the approximate formula

$$\frac{2\pi x}{p}=\log\left(\frac{1}{1+t}\right)-1$$

$$\text{or, } t=-1+e^{-\left(\frac{2\pi x}{p}+1\right)}$$

In the actual case with which we are at present concerned, x is not less than 5 (the width of the guard ring) and p is not greater than 2.5, hence $\frac{2\pi x}{p}$ is not less than 12.56. Hence, as a second approximation we have

$$\log\frac{1}{1+t}=\frac{2\pi x}{p}+1-e^{-\left(\frac{2\pi x}{p}+1\right)}$$

$$\therefore E=\frac{V}{4\pi p}\left[x+\frac{p}{2\pi}-\frac{p}{2\pi}e^{-\left(\frac{2\pi x}{p}+1\right)}\right]$$

Hence, the charge on the strip included between $x=x_1$ and $x=x_2$ is

$$E=\frac{V'}{4\pi p}\left[x_2-x_1+\frac{p}{2\pi}\left\{e^{-\left(\frac{2\pi x_1}{p}+1\right)}-e^{-\left(\frac{2\pi x_2}{p}+1\right)}\right\}\right]$$

That is, the effective width of the strip is greater than its true width by an amount

$$\delta l=\frac{p}{2\pi}\left\{e^{-\left(\frac{2\pi x_1}{p}+1\right)}-e^{-\left(\frac{2\pi x_2}{p}+1\right)}\right\}$$

The actual increase in the capacity of our circular plate will be less than that obtained by assuming that this entire excess of charge is situated on the extreme edge of the collector plate. Hence, the effect of the finite width of the guard ring is to increase the capacity of the plate by an amount which is less than

$$\Delta_2 C=\frac{R\delta l}{2p}\left(1+\frac{1}{4}\frac{p}{R}\right)$$

where the factor $\left(1+\frac{1}{4}\frac{p}{R}\right)$ is J. J. Thomson's correction for curvature. Hence the relative increase in capacity will be less than

$$\frac{\Delta_2 C}{C_0}=\frac{2\delta l\left(1+\frac{1}{4}\frac{p}{R}\right)}{R}$$

In the present case the width of the guard ring is 5 cm, R is 10 cm, hence

$$\frac{\Delta_2 C}{C_0}=\frac{p\left\{e^{-\left(\frac{10\pi}{p}+1\right)}-e^{-\left(\frac{30\pi}{p}+1\right)}\right\}}{10\pi}\left(1+\frac{p}{40}\right)$$

$$=0.0118\,p\left\{e^{-\frac{31.4}{p}}-e^{-\frac{94.2}{p}}\right\}\left\{1+0.025\,p\right\}$$

Giving p the greatest value used in this work (2.5 cm), we find

$$\frac{\Delta_2 C}{C_0}=1.4\times 10^{-7}$$

Hence the width of the guard ring is more than sufficient for the highest possible accuracy.

38. CORRECTION FOR GUARD RING GAP AND FOR DISPLACEMENT OF THE PLATE FROM THE PLANE OF THE GUARD RING.

The capacity of a circular plate condenser is[12]

$$C=\frac{\left[R+\delta l\left(1+\frac{1}{4}\frac{p}{R}\right)\right]^2}{4p}$$

Where R is the radius of the plate, p is the distance between the plates and δl is the edge correction when R is infinite.

On page 517 (equation 25) we found the following expression for δl for the case in which the distance between the plates is different on the two sides of the gap:

$$\delta l=\frac{w}{2}+\frac{1}{\pi}\left[p\log\frac{(p+q)^2+w^2}{4p^2}+(q-p)\log\frac{(p+q)^2+w^2}{A}-w\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]$$

$$=\frac{w}{2}+\Delta l$$

where p is the distance between the plates on the collector plate side of the gap, q is the distance between the plates on the guard ring side of the gap, w is the width of the gap, and

$$A\equiv\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}$$

For abbreviation we write

$$\Delta l\equiv\frac{1}{\pi}\left[p\log\frac{(p+q)^2+w^2}{4p^2}+(q-p)\log\frac{(p+q)^2+w^2}{A}-w\sin^{-1}\frac{w^2+q^2-p^2}{A}\right] \quad (28)$$

Hence,

$$C=\frac{\left(R+\frac{w}{2}\right)^2}{4p}\left\{1+\frac{wp}{8R^2}+\frac{\Delta l}{R}\left(1+\frac{1}{4}\frac{p}{R}\right)\right\}^2$$

[12] Maxwell, Electricity and Magnetism, Vol. I, § 201.

which may be written

$$C=\frac{\left(R+\frac{w}{2}\right)^2}{4d}$$

We shall call the quantity d the effective distance between the plates. If p_1 is the actual distance between the plates, the effective distance is

$$d_1=p_1\left\{1-\frac{wp_1}{4R^2}-\frac{2(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)\right\} \text{ approximately.}$$

Increasing the distance between the plates until it is equal to p_2

$$d_2=p_2\left\{1-\frac{wp_2}{4R^2}-\frac{2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)\right\}$$

$$\therefore d_2-d_1=p_2-p_1+\left[\frac{2p_1(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)+\frac{wp_1^2}{4R^2}-\frac{2p_2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)-\frac{wp_2^2}{4R^2}\right]$$

By the method of procedure adopted in this work we calculate a_1, measure p_2-p_1, and assume $d_2=d_1+p_2-p_1$. Hence, the relative amount which must be added to the assumed capacity at the distance p_2 to make it equal to the true capacity is

$$\frac{\Delta_2 C}{C_0}=\left\{\frac{2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)+\frac{wp_2}{4R^2}\right\}-\frac{p_1}{p_2}\left\{\frac{2(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)+\frac{wp_1}{4R^2}\right\} \quad (29)$$

We may write this in the form

$$\frac{\Delta_2 C}{C_0}=F(p_2)-\frac{p_1}{p_2}F(p_1)$$

where

$$F(p)\equiv\frac{2(\Delta l)}{R}\left(1+\frac{1}{4}\frac{p}{R}\right)+\frac{wp}{4R^2}$$

If $(q-p)$ is not too great, we can expand $F(p)$ in ascending powers of $(q-p)$. Omitting all after the term of the first power, we have

$$F(p)=F_0(p)+(q-p)F_0'(p)+\ \ldots$$

$$=\frac{2}{R\pi}\left[p\log\frac{4p^2+w^2}{4p^2}-w\sin^{-1}\frac{w}{\sqrt{4p^2+w^2}}\right]\left(1+\frac{1}{4}\frac{p}{R}\right)+\frac{wp}{4R^2}$$

$$+\frac{q-p}{R\pi}\left[\log\frac{4p^2+w^2}{w^2}\right]\left(1+\frac{1}{4}\frac{p}{R}\right)$$

and

$$\frac{\Delta_3 C}{C_0}=F_0(p_2)-\frac{p_1}{p_2}F_0(p_1)+(q-p)\left[F_0'(p_2)-\frac{p_1}{p_2}F_0'(p_1)\right] \qquad (30)$$

The values of $F_0(p)$ and of $F_0'(p)$ for several values of p, and for $w=0.0179$, and $R=10$ are given in Table XIV. These values apply directly to the results obtained from plate B.

TABLE XIV.

Correction for Displacement of Guard Ring.

p	$F_0(p)$	$F_0'(p)\times 10^{-4}$
0.05	-99×10^{-6}	11.1×10^{-6}
0.15	-28 "	18.0 "
0.25	-9 "	21.4 "
0.55	$+15$ "	26.6 "
1.05	$+42$ "	31.1 "
2.05	$+89$ "	36.4 "
2.55	$+112$ "	38.3 "

If $(q-p)$ is expressed in microns, the numbers in the third column are the values of $F_0'(p)$ to be substituted directly in equation 30. Hence, even if $(q-p)$ were so small as 5 μ the assumed capacity for p_2 equal to 0.25 would be too small by 11 parts in 100,000, if p_1 were equal to 0.05; for p_2 equal to 2.55 it would be too small by 30 in 100,000.

An accuracy of adjustment approaching this was attained only in the last few determinations. For all the former work the accuracy of the adjustment is unknown and can not be satisfactorily determined; the relative displacement of the collector plate and the ring was probably large.

[The electromagnetic measurements and the concluding portion of this article will be given in the next number of this Bulletin].

WASHINGTON, May 20, 1907.

A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

[Continued.]

By E. B. Rosa and N. E. Dorsey.

VII. DETERMINATION OF CAPACITIES IN ELECTROMAGNETIC MEASURE.[13]

In the electromagnetic measurement of the capacity two distinct methods were employed, viz: (1) The method described by Maxwell;[14] (2) the differential galvanometer method. For the latter method connections were so made that we could readily measure either the charging or the discharging current.

39. MAXWELL'S BRIDGE METHOD.

As is well known this method consists in replacing one of the resistances in a Wheatstone bridge by a condenser and a suitable device for rapidly and regularly charging and discharging the condenser. Maxwell originally suggested a periodic reversal of the terminals of the condenser, but this is not so good, mechanically, as the periodic charging and discharging by touching one terminal of the condenser P (Figs. 29, 30) alternately to R and S. Then, by a suitable adjustment of the frequency of the charging, and of the magnitude of the resistances of the net, a balance of the galvanometer can be obtained. When this is done the relation between the electromagnetic capacity and the resistance will be[15]

[13] We are under obligations to Dr. F. W. Grover and Mr. H. D. Babcock for assistance in some of the experimental work, and to Mr. G. W. Vinal and Miss G. C. U. McDermut for assistance in preparing the results for publication.

[14] Electricity and Magnetism, Vol. II, § 776.

[15] J. J. Thomson, Phil. Trans., **174**; 1884. Proc. Roy. Soc. **35**, p. 346; 1883. Rosa and Grover, this Bulletin, **1**, p. 153; 1905.

$$C=\frac{a\left[1-\frac{a^2}{(a+c+g)(a+b+d)}\right]}{ncd\left[1+\frac{ab}{c(a+b+d)}\right]\left[1+\frac{ag}{d(a+c+g)}\right]}=\frac{aF}{ncd}$$

where

$$F\equiv\frac{1-\frac{a^2}{(a+c+g)(a+b+d)}}{\left[1+\frac{ab}{c(a+b+d)}\right]\left[1+\frac{ag}{d(a+c+g)}\right]}$$

The letters stand for the values of the resistances (in international ohms) of the branches against which they are placed in Figs. 29, 30; n is the number of charges of the condenser per second. F is a correction factor differing but little from unity, in our work only a few parts in 100,000.

In the case of guard ring condensers it is necessary that the guard ring be charged to the same potential as the main condenser. This

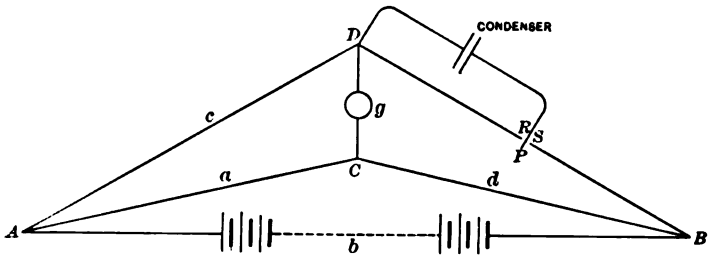

Fig. 29.—*Maxwell Bridge for Measuring the Capacity of a Condenser.*

The points P, R, S are joined to B_3, B_2, B_1, respectively, of the commutator, Figs. 32 and 32a.

condition is readily obtained by constructing, as shown in Fig. 30, a second Wheatstone net exactly duplicating the main net and containing the same battery; for, if the two nets contain the same resistances branch for branch, the maximum potential of D′ must be the same as the maximum potential of D, and this maximum potential is the potential to which the condenser and ring are charged, provided neither of the charging points P and P′ breaks contact with the corresponding charging (or discharging) points R, R′ (or S, S′) before *both* condenser and ring are fully charged (or discharged). This condition is easily fulfilled. It is also necessary,

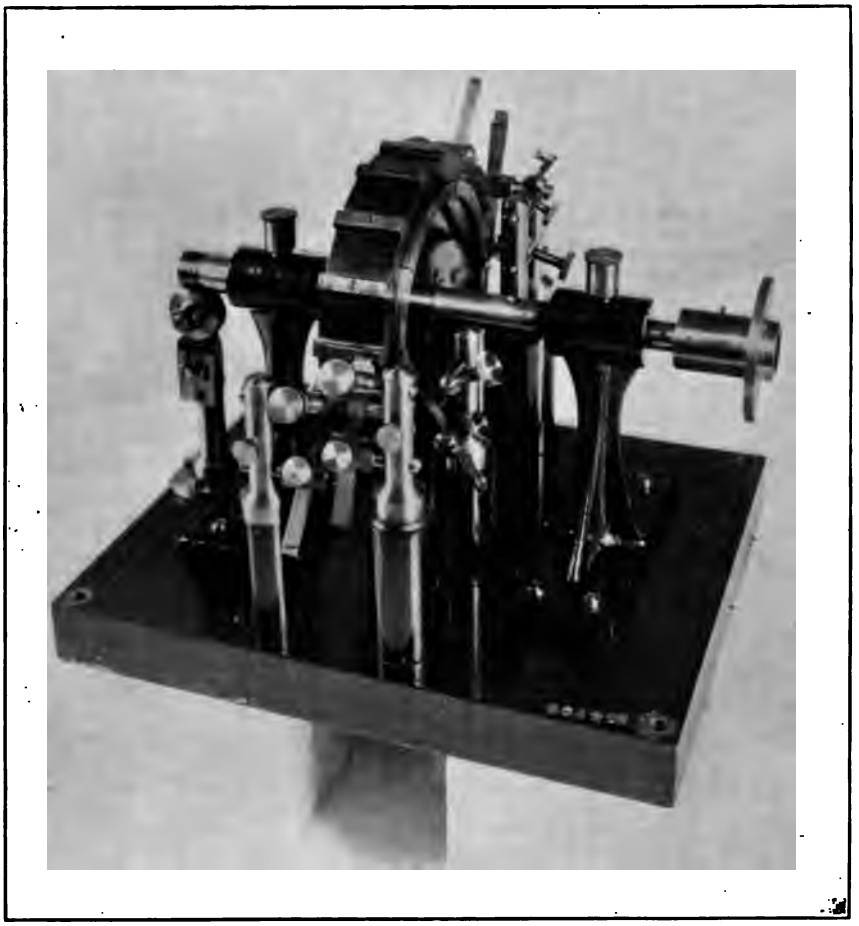

Fig. 31.—*Rotating Commutator.*

for exact work, for the continuous plate of the condenser to be connected to the point B instead of D.

The charging and discharging of the condenser is effected by means of a rotating commutator similar to the commutator described in this Bulletin, vol. 1, p. 172, except that there are sixteen contact pieces, and the ring of brass to which the contact pieces are connected is divided into sixteen segments T_1, T_2, etc., insulated from one another so that but one segment and contact piece is charged at a time. This materially lessens the capacity of the commutator, and allows the guard ring of the condensers to be charged by the

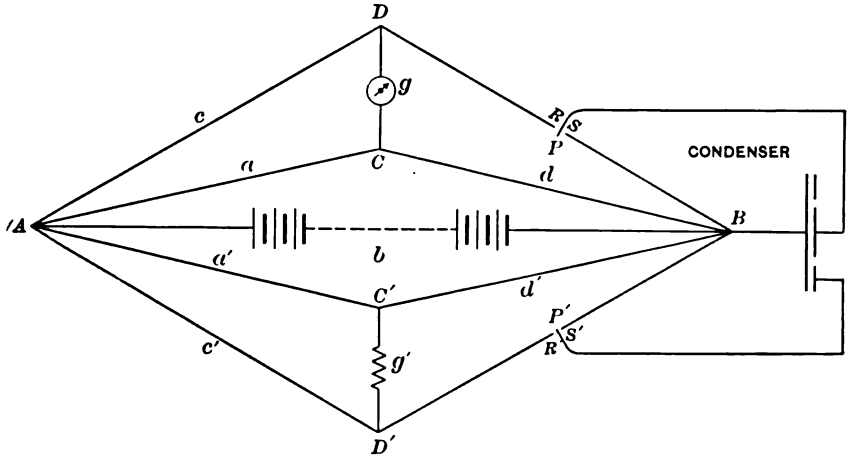

Fig. 30.—*Double Maxwell Bridge, for Measuring the Capacity of a Guard Ring Condenser.*

The corresponding arms a, a′; c, c′ etc., have identical resistances, and the potentials of D, D′, and C, C′ are therefore the same. P′, R′, S′ are joined to the second set of brushes of the commutator B_6, B_5, B_4 of Fig. 32a.

same commutator as that employed for the main condenser, the charging of the ring being effected by a second set of three brushes, B_4, B_5, B_6, placed diametrically opposite those which serve for the main condenser. The insulation across the commutator was frequently tested and was always found to be as nearly perfect as could be desired; its resistance was of the order of a million megohms. The contact pieces S_1, S_2, etc., of the commutator are of phosphor-bronze, and the brushes B_1, B_2, etc., are of thin brush copper clamped between springs reaching to within about 4 mm of their tips; steadying screws with dampers reduced the vibration. As shown in Figs. 31, 32, and 32a, the phosphor-bronze segments project

beyond the surface of the ebonite disk of the commutator so that the tips of the brushes B_1, B_2, etc., are *air insulated*, except when in contact with the metal segments. The commutator was ordinarily driven at the rate of 1,200 to 1,700 revolutions per minute. A counter C made contact and recorded every fiftieth revolution on the chronograph.

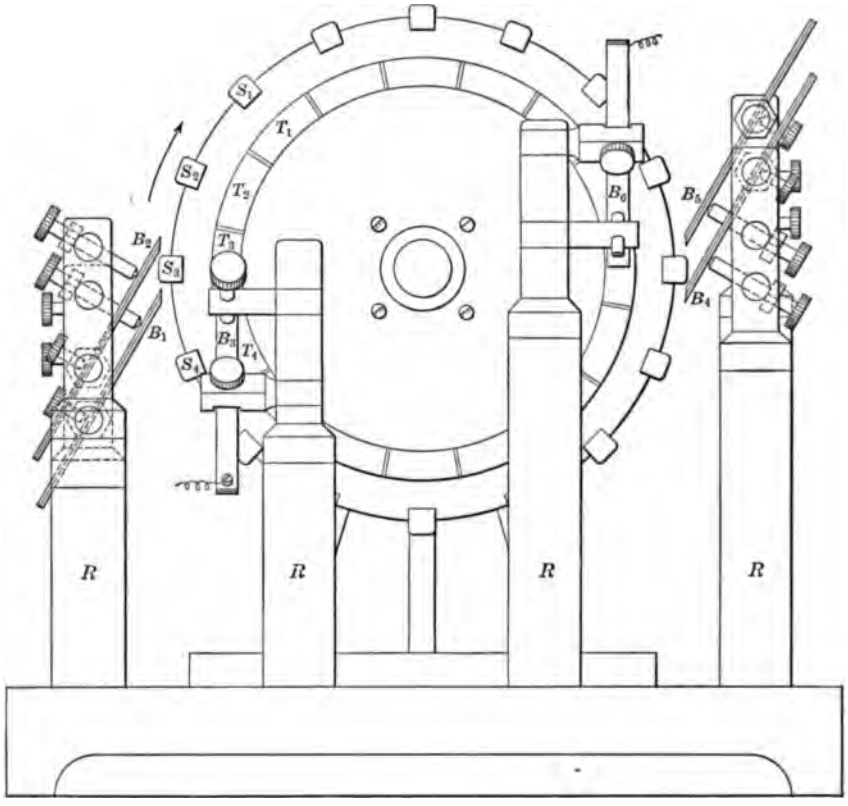

Fig. 32.—*Rotating Commutator for Charging and Discharging the Condensers.*

The six metal brushes are mounted on six ebonite columns, three on each side. The condenser is joined to B_3, and then to T_3 and the segment S_3 when the commutator is in the position shown in the figure. The condenser is charged when S_3 touches the brush B_1 and discharged when it touches B_2. It is charged again an instant later when S_4 touches B_1, etc. The guard ring of the condenser is joined to B_6, and is similarly charged and discharged by the brushes B_4 and B_5 simultaneously with the charging and discharging of the main condenser. The brushes are carefully adjusted so that the interval between charging and discharging shall coincide as nearly as possible for the condenser and its guard ring.

40. DIFFERENTIAL METHOD.

In this method the charging or the discharging current is passed through one coil of a differential galvanometer, while through the other coil is passed a steady current shunted off from a portion of a high resistance which permanently connects the terminals of the

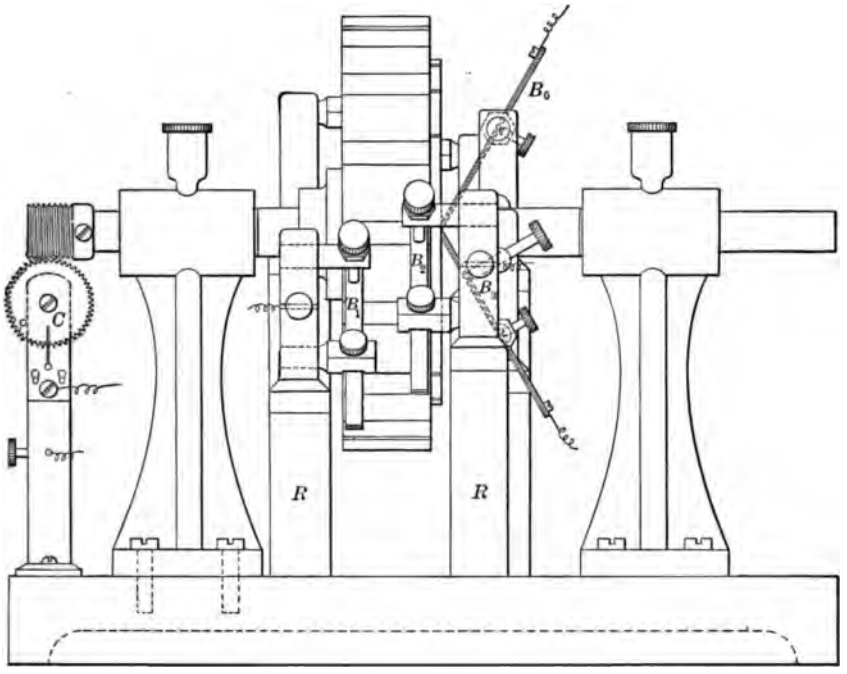

Fig. 32a.—*Rotating Commutator; Side View.*

battery employed to charge the condenser. The continuous lines in Fig. 33 show the connections in case we wish to send the discharge current through the galvanometer. If we wish to send the charging current through it, we disconnect BE and connect B′E′. If the condenser has a guard ring the connections for measuring the discharge current are shown by the continuous lines in Fig. 34. To measure the charging current we disconnect JK and connect J′K; and interchange the connections at the posts S and R also at the

posts S′ and R′. If as before we denote the resistances of the various branches by the small letters written near them, and call E the emf.

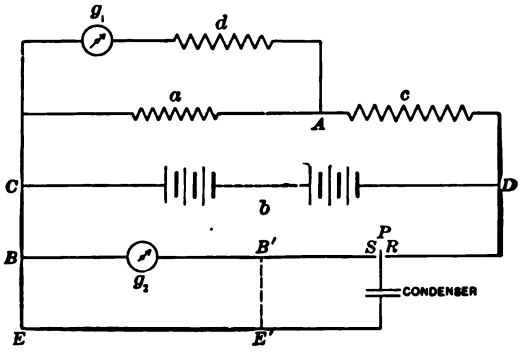

Fig. 33.—*Arrangement of Resistances, Galvanometer Coils, etc., for Measuring the Capacity of a Condenser by the Differential Method.*

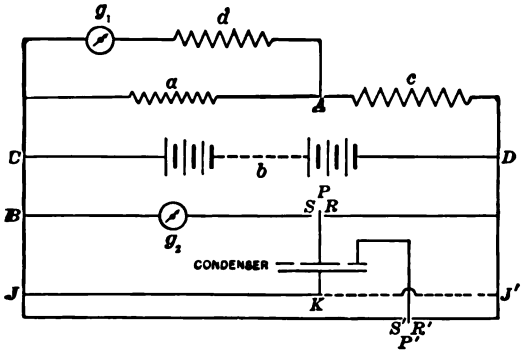

Fig. 34.—*Arrangement of Resistances, Galvanometer Coils, etc., for Measuring the Capacity of a Guard Ring Condenser by the Differential Method.*

Same as Fig. 33, except for connections of guard ring to second set of commutator brushes, etc.

of the battery, we find the following equations for the differential method:

$$\text{Current through AD} = i = \frac{E}{c + \frac{a\,(d+g_1)}{a+d+g_1}} = \frac{E(a+d+g_1)}{c\,(a+d+g_1)+a\,(d+g_1)}$$

$$\text{Current through } g_1 = i_1 = \frac{ai}{a+d+g_1} = \frac{Ea}{cd\left[1+\frac{a+g_1}{d}+\frac{a}{cd}\left(d+g_1\right)\right]}$$

$$= \frac{Ea}{cd\left[1+\frac{g_1}{d}+a\left(\frac{1}{d}+\frac{1}{c}+\frac{g_1}{cd}\right)\right]}$$

$$= \frac{Ea}{cd[1+A_2]}$$

where

$$A_2 \equiv \frac{g_1}{d}+a\left[\frac{1}{d}+\frac{1}{c}+\frac{g_1}{cd}\right]$$

The current through $g_2 = i_2 = nCE$, n being the number of charges per second. Hence for balance when $i_1 = i_2$

$$nC = \frac{a}{cd(1+A_2)}$$

or

$$C = \frac{a}{ncd(1+A_2)}$$

This formula is strictly analogous to the one obtained for the bridge method. However, the quantity A_2 is here not small, but may amount to several per cent of the whole. The galvanometer coils were shunted so as to make the system aperiodic and at the same time to make the coils magnetically equivalent, so that, when a considerably larger current than any used in measuring capacities is sent through the two coils in opposite directions, the deflection produced is too small to be observed. This equality test was made frequently. The resistances g_1 and g_2 are the effective resistances of the coils of the galvanometer with their respective shunts. The resistances g_1 and g_2 were measured whenever the galvanometer was used.

A great advantage of the differential method is that it gives a direct check against the insulation of the armatures of the condenser. If there is any appreciable leakage between the plates of the condenser, either through their supports or by way of the commutator and conductors leading to it, then the capacity of the condenser as measured by the use of the charging current must of necessity be greater

than the capacity as measured by means of the discharging current. No certain difference of this kind was ever noticed. This method, however, does not give a check against leakage from the brush S to

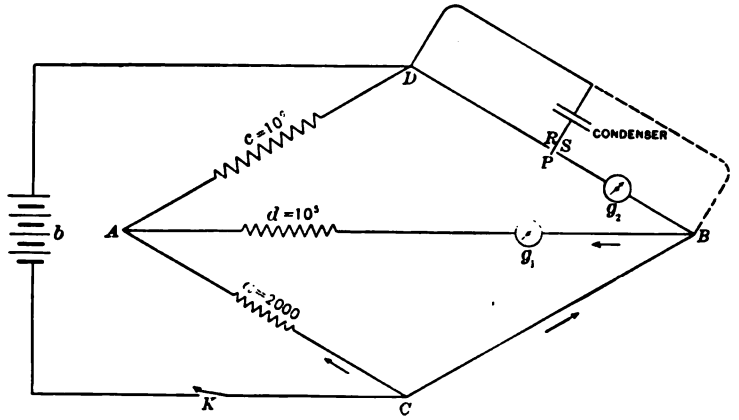

Fig. 35.—*Arrangement of Resistances as in Fig. 33 for the Differential Method, Showing the Resemblance to the Maxwell Bridge.*

The arm BC has zero resistance.

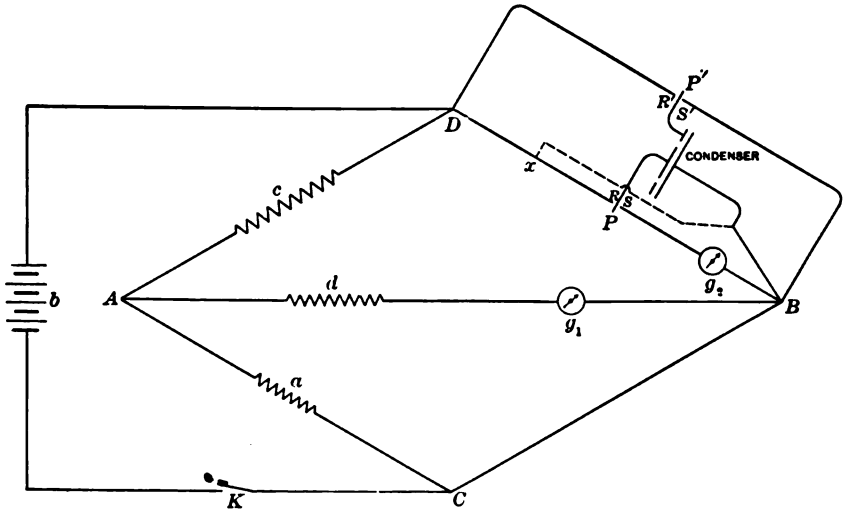

Fig. 36.—*For Measuring Guard Ring Condenser by Differential Method.*

Same arrangement of resistances as shown in Fig. 34.

the brush R. The resistance between these points was, however, frequently measured and was always of the order of a million megohms.

Another advantage of this method is that the resistance in the charge or discharge circuit is always rather small, being merely that of the galvanometer coil g_2, which is about 80 ohms. It is the same whatever the capacity, so that the duration of contact of the brushes in this method might be shorter, if desired, than in the bridge method. However, they were always adjusted by the same criterion as described on page 552.

The connections for the differential method are also represented in Figs. 35 and 36, which are exactly equivalent to Figs. 33 and 34.

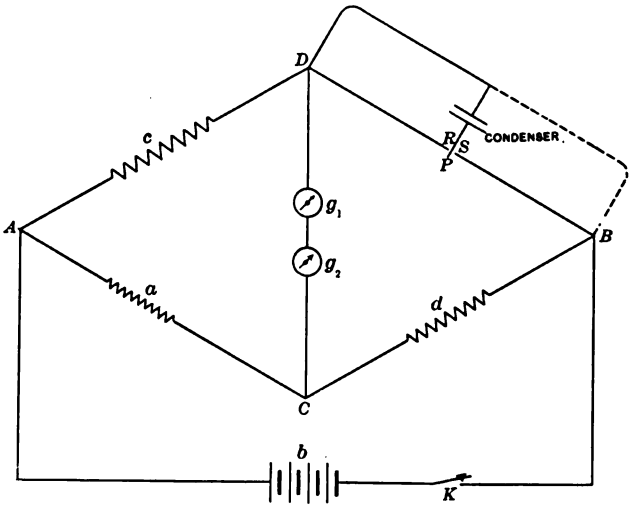

Fig. 37.—*Showing how the Maxwell Bridge Arrangement is Obtained by Rearranging the Resistances and Galvanometer Coils of the Differential Method, Fig. 35.*

To change the connections shown in Fig. 35 into the proper connection for the bridge method we have only to (1) interchange the battery and galvanometer (putting both halves of the galvanometer in series) and (2) put the resistance d' in the arm CB (see Fig. 37). We now have a Maxwell bridge consisting of the same resistances, and it will be balanced when a small change is made in a, corresponding to the difference in the correction factors, F in the one case and $1+A_2$ in the other. This change is so quickly made that it is very convenient to compare measurements by the two methods, especially in the case of the spherical condensers where, owing to the absence of a guard ring, a single bridge is used.

41. COMPLETENESS OF CHARGE.

If the duration of contact of the commutator brushes is not sufficient, the condenser will not be completely charged or completely discharged, hence, the measured capacity will be less than the true capacity in the ratio of $(1-e^{-\frac{t_1}{CR_2}})$ to 1. Where t_1 is the duration of contact of the brushes, C is the capacity, and R_2 is the resistance of the network of conductors joining the points B and D of the bridge.[16] If, however, d is very large (in our case it is 10^5 ohms), then R_2 may be replaced by its approximate value[17] $R'_2=b+\frac{c(a+g)}{c+a+g}$

For the large sphere and the high resistance galvanometer $C=5.5\times10^{-11}$ farads, $a<3{,}500$ ohms, $b=100$ ohms $c=10^6$ ohms, $g=1{,}000$ ohms, $\therefore R'_2C<2.5\times10^{-7}$ seconds.

The highest speed ever reached during the work was 1,960 revolutions per minute, the circumference of the commutator over contact pieces is 40 cm, the width of each contact piece is 0.5 cm; hence, the time required for a contact piece to pass any given point is $t=\frac{0.5}{40}\times\frac{60}{1{,}960}=3.8\times10^{-4}$ seconds. If the brush makes contact over $\frac{1}{n}$ of the total width of a contact piece $t_1=\frac{t}{n}=\frac{3.8}{n}\times10^{-4}$

Hence, $$e^{-\frac{t_1}{CR_2}}=e^{\frac{-3.8\times10}{2.5n\times10^{-7}}}=e^{-\frac{1.5\times10^3}{n}}$$

If we allow an error of so much as 1 in 10,000 we can determine the maximum allowable value of n from the equation

$$e^{-\frac{1.5\times10^3}{n}}=10^{-4}$$
$$\therefore n=160$$

Hence, for this case the error due to this cause will amount to only 1 in 10,000 even when the brushes never make contact over more than $\frac{1}{160}$ of the total width of a commutator segment. By means of the tests with the milammeter, as described on page 552, it was estimated that at no time, when the action of the commutator was

[16] See Maxwell, Elec. and Mag., Vol. II, § 777.
[17] This Bulletin 1, p. 169; 1905.

regarded as good enough to give reliable results, did the brushes make contact over less than 0.75 of the width of the segment, and usually (owing to the finite length of the portion of the brush making contact) the contact extended over the entire width of the segment.

For the largest capacity practically obtainable with the plate condensers used, we have $C = 119 \times 10^{-11}$ farads, $a = 14{,}000$, $g = 1{,}000$, $c = 0.250 \times 10^{6}$, $b = 100$; $\therefore CR_2 = 1.7 \times 10^{-5}$. Hence for an accuracy of 1 in 10,000 we must have n no greater than is given by the equation

$$e^{-\frac{3.8\times10^{-4}}{1.7n\times10^{-5}}} = 10^{-4}$$

$$\therefore n = 2.44$$

Hence, even in this extreme case, which was never reached except in the preliminary work, the margin of safety would appear to have been sufficient.

All of this applies directly to the case in which the brushes are working smoothly and possess but little free vibration. This condition can always be obtained by properly adjusting the springs between which the copper brushes are held and the angles at which the segments meet the brushes. Hence for condensers without guard rings this correction can be made vanishingly small. Unfortunately, however, in the case of guard-ring condensers it is often impracticable to get *both* sets of brushes running smoothly simultaneously and still have them in proper phase relation. Under these conditions the brushes are set in rather violent free vibration, which renders the duration of contact uncertain and gradually bends the brushes so that the contact becomes worse and worse. This condition can be improved, but not completely eliminated, by placing soft rubber dampers on the backs of the brushes and extending nearly to their tips. These rubbers are gilded so as to eliminate any irregular creeping of electricity over them. Having got the brushes adjusted as well as possible, they are screwed down against the commutator until they seem to make as long a contact as is possible without harmful vibration. The bridge is then balanced, after which an extra resistance of 10,000 or 20,000 ohms is introduced seriatim in the branches DR, SB, BS′, R′D′. If the balance

is not affected by this process we are justified in concluding that the brushes make sufficiently good contact. If the balance is affected by this process the brush or brushes at fault are readjusted. Having thus got the brushes into what is known to be a good adjustment, it is possible to obtain an empirical and more expeditious method of testing the functioning of the brushes. This consists in placing a milammeter and storage cell in series with P and R, D and B being disconnected from R and S, respectively. The reading of the milammeter gives a measure of the average duration of contact, and the constancy of its reading is an index to the regularity of the contact. Similarly the other three contacts are tested. After a little experience in this way under conditions which are known to be good, or bad, as the case may be, one learns what reading of the ammeter to expect under different conditions, after which it serves as a very ready and satisfactory method for determining the behavior of the brushes.

42. RESISTANCES.

The resistances employed were as follows: In the arm *a*, at various times were used, (1) a box by Otto Wolff, No. 2470; resistances range from 0.1 ohm to 50,000 ohms, the coils are of manganin, are wound noninductively upon metal spools, and were oil immersed. (2) An exactly similar box, No. 2471. The coils of these two boxes are shortcircuited, when desired, by conical plugs in the usual way. (3) A dial box, also by Wolff, No. 3087, having a range from 0.1 ohm to 10,000 ohms. These coils are similar to the coils of the other boxes, but were not oil immersed.

Arm *c* always consisted of one or other of two similar boxes by Siemens and Halske, No. 15657 and No. 16228. Each consists of a series of ten manganin coils, nominally of 100,000 ohms each. The coils are embedded in paraffin into which holes are drilled for the insertion of thermometers. The total resistance of No. 15657 is approximately 1,000,880 international ohms at 20°.0 C, and that of No. 16228 is 1,001,180 international ohms at 20°.0 C.

For the larger capacities the megohm in *c* has its two halves connected in parallel, making the resistance nominally 250,000 ohms.

Arm *d* consisted of one of the following: (1) A box by Otto Wolff, No. 2620, consisting of a series of ten manganin coils of approxi-

mately 10,000 ohms each. The coils are always oil immersed. (2) An exactly similar box, No. 2621, also oil immersed.

Thus, the resistance of *d* is always nominally 100,000 ohms, and the resistance of *c* is always nominally 1,000,000 or 250,000 ohms.

The resistance of *a* is varied so as to balance the bridge.

These coils above 500 ohms were intercompared almost daily when in use, and comparisons were made at frequent intervals with the standards of this Bureau. The intercomparison was performed as follows: The 10,000 ohm combination, obtained by connecting the coils of the megohm in parallel, was compared with the 10,000 ohm combination obtained by connecting the first nine coils of the 100,000 ohm box in square array, three in series and three such groups in parallel; then the coils in the 100,000 ohm box were connected in parallel and the 1,000 ohms thus obtained was compared with the 1,000 ohm coil of the box in arm *a*. The other coils of *a* were compared with the 1,000 ohm coil in the usual manner. Thus, we get all of the resistances in terms of the 1,000 ohm coil in the box in arm *a*. This and other coils are frequently compared with the standards. This method of comparison is justified by the fact that all the coils in any one of the decade boxes considered have nearly the same value; the error thus introduced in the case of these boxes does not amount to more than 2 parts in 1,000,000.

43. STANDARDS OF RESISTANCE.

The standard resistance employed in this work is the mean value of the resistances of 11 one-ohm and 7 tenth-ohm coils belonging to this Bureau. Groups of these coils have recently and at various times in the past been compared with the standards of the Physikalisch-Technische Reichsanstalt, so that we feel confident that the value of this mean is known in terms of the international ohm with an accuracy in excess of that required for the present work.

44. SEASONAL CHANGES OF THE RESISTANCES.

By the process of intercomparison of the various coils and the frequent comparison of certain of them with the standards as just described, we obtain data from which, after proper temperature corrections have been applied, we are able to construct curves showing the relation between the corrections to be applied to the different

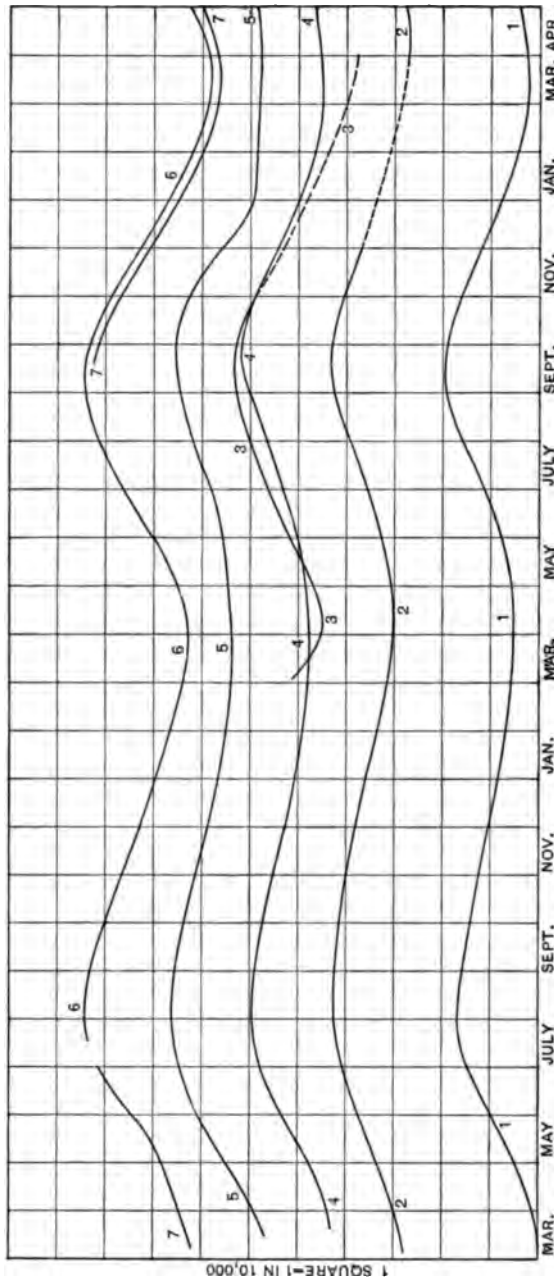

Fig. 38.—*Showing the Seasonal Changes of the Resistances Employed in the Measurement of the Capacities of the Condensers.*

The resistances were of manganin and were kept submerged in oil. All measurements after being reduced to a common temperature show maximum values in the summer and minimum in the winter. These changes have recently been found to be due to atmospheric moisture.

coils at some constant temperature (say 20°.0 C) and the time. Doing this we obtain the very surprising and interesting group of curves shown in Fig. 38. The ordinates represent the correction in parts per 10,000 to be added to the nominal value of the coil in order to obtain its true resistance in terms of our standard ohm, the origin being arbitrary. The abscissæ represent the time, each division corresponding to one month. It will be noticed that every curve has a minimum between February and April, then rises gradually to a maximum in July and August, then decreases to a minimum again the following February and April. This unanimity is so much the more surprising when we consider the great difference in the coils under study. The behavior of the

1,000-ohm coil of No. 2470 is shown in curve No. 1;

1,000-ohm coil of No. 2471 is shown in curve No. 2;

1,000-ohm coil of No. 3087 is shown in curve No. 3;

Combination of (500+200+200+100) ohm coils of No. 2470 is shown in curve No. 4;

Combination of (500+200+200+100) ohm coils of No. 2471 is shown in curve No. 5;

Ten 10,000-ohm coils of No. 2620, connected in parallel, is shown in curve No. 6;

Ten 10,000-ohm coils of No. 2621, connected in parallel, is shown in curve No. 7.

Boxes No. 2470 and No. 2471 are probably of the same age and have been subjected to the same treatment, except that No. 2471 has been at constant humidity since December, 1906. The coils of these boxes have been in oil since February, 1905. Box No. 3087 is a new box of a different type from No. 2470. It was purchased in the winter of 1905–6; its coils are not oil immersed. Boxes No. 2620 and No. 2621 are probably of the same age and have undergone similar treatment; the coils are oil immersed. Thus four of the seven curves are entirely unrelated to one another. These are

I. Curves 1 and 2, behavior of certain 1,000-ohm coils in oil;

II. Curve 3, behavior of a different 1,000-ohm coil in air;

III. Curves 4 and 5, behavior of coils under 1,000 ohms in oil;

IV. Curves 6 and 7, behavior of coils of 10,000 ohms in oil.

Throughout this time the boxes have all been kept at laboratory temperatures which have fluctuated from about 15° to 30° C, though

the temperature was rarely under 19° or over 27° C; they have not been heavily loaded, and in no sense can their treatment, other than by the weather, be said to have been of a periodic character. The variations can not be due to leakage produced by moisture condensed upon portions of the box, for the resistances of the coils are highest during July and August, when the humidity in this region is the greatest; and it is much too great to be explained by an uncertainty in the temperature coefficients.

It is probably due to strains produced by the expansion of the shellac, in which the coils are embedded, as moisture is absorbed, the oil not protecting the coils completely, even when they are completely submerged. This question has been under investigation in this laboratory for some time and will form the subject of another paper. These curves emphasize the necessity of very frequent comparisons of resistances with the standards when an accuracy of 1 in 10,000, or better, is desired.

The temperature coefficients of the other coils employed were not determined, but they were almost daily compared at laboratory temperatures with the coils of which we know the coefficients. This enables us to correct our results for any variations in the resistances used.

45. GALVANOMETERS.

Two galvanometers, both of the D'Arsonval type, were used. One was a 1,000-ohm galvanometer by Sullivan. It had a sensibility of 1,000 mm, at 1 meter scale distance, per microampere. Its period is about nine seconds, damping on open circuit is very slight, and its zero is very constant. It was used at a scale distance of 2.5 meters, and was shunted by a noninductive resistance of 10,000 ohms. This was just sufficient to make the instrument aperiodic. Its inductance, including the leads, was found to be 66 millihenrys; hence its time constant is negligible as compared with the duration of contact of the commutator brushes.

The other galvanometer was a fine differential D'Arsonval made by the Weston Electrical Instrument Company. When the coils are connected in series, it has a sensibility equal to that of the Sullivan. Each coil has a resistance of about 82 ohms, its period is about five seconds, its damping is slight, and its zero is extraordinarily con-

stant. When each coil is shunted with 2,000 ohms the instrument is nearly aperiodic. The two coils were very nearly equal in their effects, and the two circuits were made equal by a proper adjustment of the shunts. The inductance of the coils is small, but was not measured.

The only remaining question to be considered is that of the division of the current between the galvanometer and its shunt. In deducing our formulæ we assume that the intermittent current from the condenser divides between the galvanometer and its shunt in the same proportion as a constant current does; that is, in inverse proportion to their resistances. We have seen that the time constants of the galvanometers are small as compared with the duration of contact of the commutator brushes, hence the intermittent current starts from zero, rises to a maximum, and falls to zero again. Hence the momentum of the field is the same at the end of the process as it is at the start; hence the integral over this period of the applied emf. must be equal to the integral over the same time of the total dissipative force in this branch. Now the only appreciable dissipative force is ir, where i is the current and r is the resistance. Hence

$$\int_0^T V dt = \int_0^T i_1 r_1 dt = \int_0^T i_2 r_2 dt$$

$$= Q_1 r_1 \qquad = Q_2 r_2$$

Hence, the quantity of electricity carried is divided between the two circuits in the same proportion as in the case of constant currents.

Here it may be mentioned that at one time both galvanometers, each separately shunted, were connected in series and used. The high-resistance galvanometer had an inductance probably twenty times that of the low-resistance galvanometer. It was found that they were simultaneously balanced, which could not be the case if the division of the current between the galvanometer and the shunt depended on the inductance of the former.

46. REGULATION AND MEASUREMENT OF SPEED.

The commutator was driven by a small motor-generator set (0.5-kw motor and a 0.3-kw alternator), which was driven by the cur-

rent from a large storage battery. On each end of the motor-generator shaft was mounted a heavy fly wheel, and the alternator carried a load composed of two 16-candlepower lamps to steady the speed. In the armature circuit of the motor was a carbon rheostat by which the speed could be very delicately controlled. At other times the speed was regulated by gentle friction applied to the rim of one of the fly wheels on the shaft of the motor. The method of procedure has already been described in this Bulletin[18]. The resistance in the bridge having been suitably adjusted, the speed of the motor was varied until a balance was obtained, as indicated by the galvanometer; then a chronographic record of the speed was obtained while the rheostat was manipulated so as to keep the galvanometer continuously as near zero as possible. Under favorable conditions it is possible to hold the speed so nearly constant that the values found for it from successive three-minute intervals of the record will differ from one another by only a few parts in 100,000.

The time was given by one or other of two chronometers which were occasionally rated by comparison with the standard Riefler clock of this Bureau. The latter was daily compared with mean solar time, as determined at the Naval Observatory. The rates of the chronometers were found to be subject to a very slight linear change, an allowance for which is made in the reductions.

In the last part of the work another method of procedure was adopted, which is more expeditious than that just outlined. As stated above, the commutator was attached to one extremity of the axis of a small motor-generator set. To the other extremity of this axis was attached a second commutator making four contacts on each brush per revolution. This commutator, which necessarily has always the same speed as the other, together with a mica condenser (0.2 mf) formed one arm of an auxiliary Wheatstone bridge. By a proper adjustment of resistances the two bridges were balanced simultaneously for a certain speed of the machine. If the capacity of the air condenser is slightly changed (the resistance in the main bridge remaining unchanged), it will be necessary to change the speed, and consequently the resistance in one of the arms of the auxiliary bridge, in order to restore the simultaneous balance of the two bridges. Except for sign, the percentage change in speed,

[18] Vol. I, p. 176.

which is equal to the percentage change in the capacity of the air condenser, is equal to the percentage change in the resistance in the auxiliary bridge, provided the changes are not too great. Hence, we may proceed as follows: With the machine running at a convenient speed, the auxiliary bridge is balanced as nearly as possible. Then the observer at the auxiliary bridge changes the speed of the motor slightly, so as to give an exact balance, and maintains the balance until a sufficiently long chronograph run has been made. This determines the speed which corresponds to the existing values of the resistances in the auxiliary bridge. While this is being done the observer at the main bridge determines the values of the resistances a_1 and a_2 in the arm a of his bridge, which will give the nearest possible balance (1) for the condenser, leads, and commutator, and (2) for leads and commutator without the condenser. Since one-tenth of an ohm is the smallest coil in arm a it is usually impossible to get an exact balance at this particular speed. Having obtained the chronograph run, the resistance of a is made a_1 and the observer at the main bridge slowly alters the resistance in the auxiliary bridge, thus causing the auxiliary observer to slowly change the speed of the machine so as to maintain a balance in his bridge; the main observer continues this process while continually watching his galvanometer, until the main bridge is also balanced. Thus a simultaneous balance is obtained, and the amount by which the resistance in the auxiliary bridge has been altered, together with the former chronograph run, gives us directly the speed at which the main bridge is balanced when a has the value of a_1. Connections are now changed so as to measure the capacity of the leads and commutator only, a is made equal to a_2, and the process is repeated. By this process a balance can be obtained in forty to sixty seconds, while two and a half or three minutes is necessary to obtain a chronograph record of suitable length; as a consequence we can pass much more rapidly from the measurement of the capacity of the condenser, commutator, and leads to the measurement of the capacity of the commutator and leads only, and back again. This reduces the chance of the commutator capacity differing appreciable in the two cases, and enables us to make in rapid succession more determinations of the capacity in each case, both of which are conducive to greater accuracy.

This method presupposes, of course, that the capacity in the auxiliary bridge remains constant, that all variations in the resistances in this bridge are known, and that the arm in which the resistance is varied has a resistance sufficiently great to give the required accuracy of setting.

If the condenser is a good mica condenser, and if its temperature is maintained constant, which can readily be done, its capacity will remain constant to the highest degree of accuracy attainable in the measurement of capacities. Added to this, however, we have the capacity of the leads and of the commutator. If the capacity of the condenser is sufficiently large, as is always the case, the variations of these small capacities will be absolutely negligible as compared with the total capacity, unless the commutator is acting very badly. This last condition is, however, easily detected and corrected.

The only thing that can cause unknown variations in the resistances during the course of a day's work is the heating by the current, which is made rather heavy in order to obtain the desired sensibility. This effect, however, is reduced by a sufficient subdivision of the higher resistances and by closing the battery circuit a little while before beginning observations. The remainder of this effect, as well as all other slight variations, may be eliminated by taking a suitable number of chronograph runs at intervals throughout the day's work.

The total resistance in the arm of the auxiliary bridge which was varied was 50,000 ohms, and usually any one setting could be determined to within 0.5 ohm when the air condenser was in the main bridge, and to 1 ohm when the commutator and leads only were being measured. The mean of from three to six independent settings was always taken for each determination of the capacity.

In order to avoid all personal bias, the resistance which was varied was divided into two parts; one was placed near the auxiliary observer and the other near the main observer. After each setting the auxiliary observer changed his portion of the resistance by an amount unknown to the main observer, who then proceeded to change his portion of the resistance so as to restore the balance. In this way every setting was independent of the preceding one.

As it is seldom that both galvanometers are simultaneously at rest, it is often desirable for the auxiliary observer to call out,

at intervals of about one second, whether the speed is fast, slow, or right, according to the indications of his galvanometer. This serves as a guide to the main observer in estimating the correctness of the balance of his bridge. If the galvanometer in the main bridge has a period that is considerably longer than that of the one in the auxiliary bridge it will integrate the current more perfectly and the process of balancing will be rendered easier.

47. THE NEW DIRECT READING CHRONOGRAPH.

In the earlier part of our work we used a chronograph[19] driven electrically and making one revolution of the drum per minute. By means of an electric contact on the shaft of the commutator every fiftieth revolution of the commutator was recorded on the chronograph, and a simultaneous record of the second-beats of a chronometer gave a means of computing with great precision the mean speed of the motor and commutator for any three-minute interval. The great labor of reading the chronograph record and computing the speed, together with the impossibility of detecting small variations in the speed until after the record had been worked up, led us to design and build a new form of chronograph which has been of great value in this and other work where the speed of a machine or the frequency of a current must be known with high precision.

The chronograph is shown in Fig. 39, and the gears and connections to the driving motor are shown in Fig. 40. The chronograph is driven through the worm and bevel gears at a speed 250 times less than the motor, which also drives the commutator and an alternating generator which provides testing current for other purposes, the frequency of which current has sometimes to be determined. The usual speed of the machine was 1,500 revolutions per minute, and the drum of the chronograph therefore made 6 revolutions per minute, or one in ten seconds. This speed could, however, be varied through wide limits, in practice from 500 to 2,000 or more per minute.

A printing magnet[20] was used instead of a pen, and once every second it printed (through a typewriter ribbon supported on posts P_1 and P_2) a dot on the paper on the drum D of the chronograph.

[19] By the Societé Genevoise.

[20] Such as used on the Rosa curve tracer.

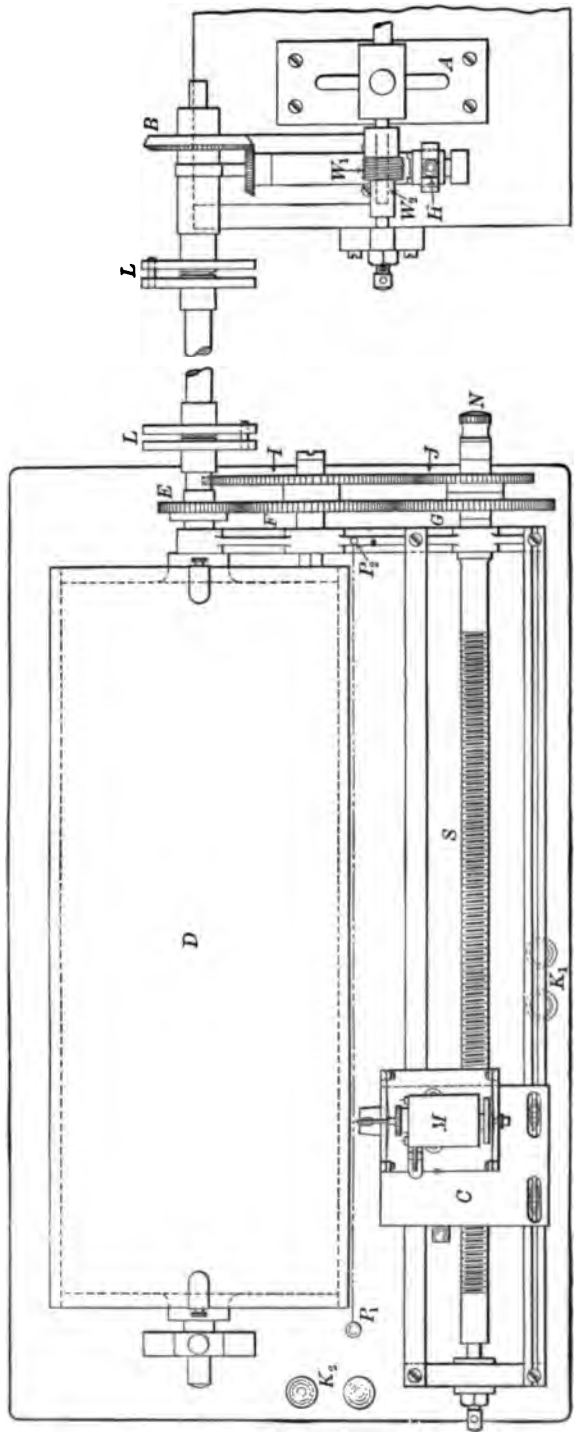

Fig. 40.—*Direct Reading Chronograph for Giving the Speed of a Generator, Commutator, or Other Machine.*

The chronograph is driven by the machine, the speed of which is to be measured, by means of the worm and gear W_1 and W_2 and the flexible coupling L L. The bevel gears B are used merely to keep the direction of the axis of the chronograph parallel to that of the driving machine. The clutch N permits the speed of the driving screws to be changed. The printing magnet M (mounted on the carriage C) prints through the type ribbon R upon the paper wrapped over the drum D. The binding posts K_1 are joined by flexible wires to the printing magnet, and the binding posts K_2 (connected to K_1 in the base) are connected to the battery and chronometer. A dot is printed once a second on the revolving drum.

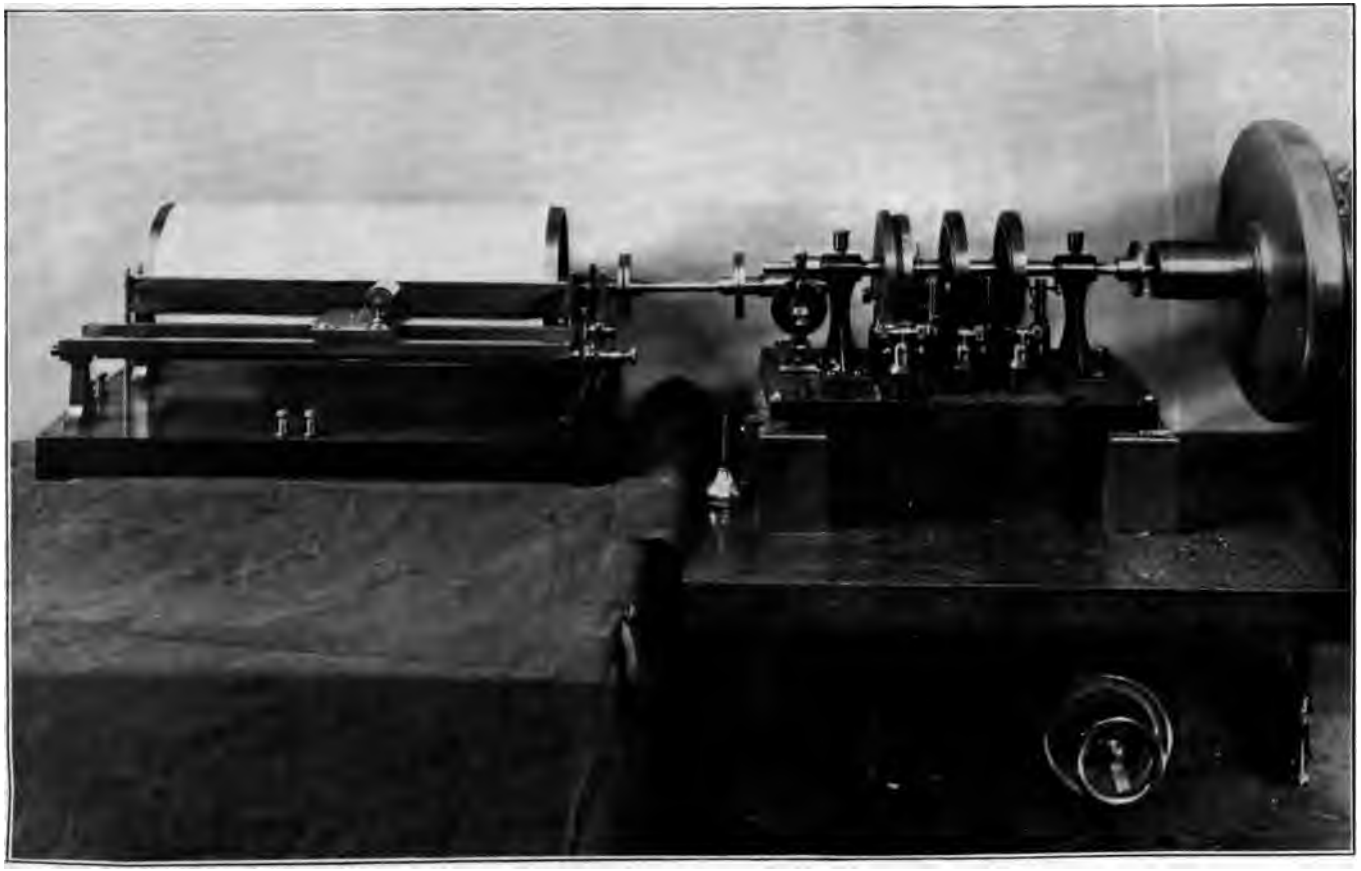

Fig. 39.—*Direct Reading Chronograph.*

At a speed of exactly 1,500 revolutions per minute there would be 10 dots in each revolution of the drum. The printing magnet was advanced by a carriage C and screw S like the pen of an ordinary chronograph, and hence there are 10 rows of dots printed, at the rate of 6 dots per minute in each row. If the speed of the motor is 1,250 instead of 1,500 there will be 12 rows of dots, printed at the rate of 5 per minute in each row, etc. The following are some of the different speeds that may be used to obtain horizontal rows of dots:

TABLE XV.

Speed	Number of rows	Number of dots per minute in each row
2,500	6	10
1,875	8	$7\frac{1}{2}$
$1,666\frac{2}{3}$	9	$6\frac{2}{3}$
1,500	10	6
1,250	12	5
1,000	15	4
750	20	3
500	30	2

If the speed is a little greater or less than 1,500, for example, the rows of dots will slope one way or the other, the angular deviation from the horizontal being a measure of the variation of the speed from the nominal speed, the direction of the slope showing whether the speed is too fast or too slow. If the speed is variable the line of dots will be wavy or irregular. One can therefore see at a glance whether the speed is regular, and soon learns to estimate the speed without stopping the chronograph (if it is steady and near the nominal value) to within a few parts in 10,000. The interval between successive rows of dots at a speed of 1,500 is 5 cm, and this corresponds to one second. Hence the record can be read much closer than the usual chronograph record.

If AB, Fig. 41, is a row of dots making an angle α with the horizontal line AC, the speed was slower than the nominal speed in that case by an amount proportional to $\tan\alpha$. The direction of rotation of the drum is indicated by the arrow.

The drum is 50 cm in circumference; hence if one revolution is

made in n seconds, the surface speed is $50/n$ cm per second. The distance $d = BC$ therefore represents a lag of the cylinder of $\Delta t = \frac{dn}{50}$ sec. Let t be the time in which the given number of revolutions have been made. As the carriage advances 1 mm with every revolution of the drum

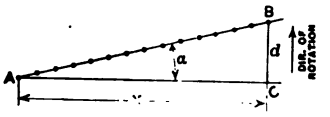

Fig. 41.

$$x = \frac{t}{10n} \text{ cm, or, } t = 10xn \text{ seconds}$$

$$\therefore \frac{\Delta t}{t} = \frac{1}{500}\frac{d}{x} = \frac{2\tan\alpha}{1000} = \text{correction to}$$

be applied to the nominal speed to obtain the actual speed for the given interval. That is, *the correction in thousandths is twice the tangent of* α. For $\alpha = 1°$, $2\tan\alpha = 0.035$, and the correction is 3.5 in 100,000. As several independent rows of dots are available for the determination of α, the error in α need not be as much as one-fourth degree when the speed is steady for a run of two or three minutes. Hence the uncertainty in the speed is less than 1 in 100,000. For longer runs the uncertainty may be still less.

Fig. 42 is a full-size reproduction of a portion of a chronograph record taken October 9, 1906. It shows 8 runs of about three minutes each, between which are records of somewhat variable speeds. At the top of the sheet is given for each run the mean amount, in parts per thousand, by which the actual speed of the machine falls short of 1,500 revolutions per minute. As explained above, these corrections are equal to the doubled tangents of the angles of slope. For example, the first run on the sheet has a mean slope of 1.8° and the correction amounts to 6 parts in 100,000. The next figure (Fig. 43) is a reduced representation of an entire chronograph sheet, which was made in comparing some condensers by alternating current, the frequency being 100 at a speed of 1,500 of the machine. The corrected frequencies for the different runs are given at the top of the sheet. In this reproduction the tails, frequently produced when the duration of contact at the relay is excessive, have been unduly magnified, and the location of the points have not been faithfully reproduced. Further than this, it gives a fair idea of the general appearance of the sheets as taken.

The separate values of the slopes are given as read, a greater number of readings being taken in some cases than in others. The advantage of knowing the speed and whether it is steady while doing the experiment, and of getting its exact value in a moment by the use of a protractor will be appreciated by anyone who has used an ordinary chronograph for such work.

48. POSSIBLE SOURCES OF ERROR DUE TO THE COMMUTATOR.

Aside from the question of leakage which has already been considered, there are five possible sources of error that may be introduced by the commutator. They are as follows:

1. The commutator increases the capacity to be measured by an amount which must be determined by a separate experiment.

2. If it cuts through a magnetic field, it may introduce a slight spurious electromotive force.

3. The friction between the segments and the brushes may introduce a differential thermal and frictional electromotive force.

4. Since the ebonite disc and the ebonite posts supporting the brushes become charged electrostatically when the commutator is running, the commutator will act like an electrostatic induction machine. Each segment (and therefore the condenser also) will acquire a certain charge, even when the battery is entirely disconnected, when the segment touches one brush, and will give it up when it touches the next. This charge adds itself algebraically to the charge given by the battery, and, if uncorrected, will lead to false results.

5. Electrostatic action of the commutator produced by the charging of the metallic brushes by the battery when the latter is in circuit.

Excepting the fourth, these sources of error occasion no serious trouble.

(1) Provided the capacity of the commutator remains constant or varies uniformly at a slow rate the first source of error occasions no inconvenience. It is necessary to eliminate the capacity of the leads from the condenser in any case, and if the commutator capacity, which is simply added to that of the leads, remains constant the capacity of the condenser (and charging wire in the case of the spheres) can be at once obtained by taking the difference between the capacity of the commutator, leads, and condenser and the

capacity of the commutator and leads alone. If the capacity of the commutator is varying slowly and uniformly, three capacity measurements only are needed, two of the capacity of the commutator and leads and one midway between the other two of the capacity of the commutator, leads, and condenser. One or other of these conditions can always be obtained by carefully adjusting the commutator brushes.

(2) In order that the second error shall be appreciable the magnetic fields normal to the two radii on which the brushes make contact must be very different. Such was not the case, however, in our experiment. The disk of the commutator lies in the plane of the earth's field and the stray field from the motor is very slight. Any slight effect of this kind that might exist would be eliminated along with the elimination of the electrostatic effect considered under the fourth head.

(3) The only condition under which the third source of error can affect the results is when the two brushes are unequally heated or make contact for unequal times. Such differences are very slight. Furthermore, the effect would be greatly enhanced by making the brushes of different materials. This was tested by making one brush of copper and the other of phosphor-bronze. The interchanging of these brushes produced no observable effect, thus showing that this source of error is so small as to be entirely masked by the next source of error to be considered. Furthermore, a small effect from this source will be eliminated along with the errors to be next considered.

(4) The fourth source of error is far greater and more troublesome than all the others taken together. It shows itself by producing a deflection of the galvanometer when the battery is entirely disconnected and the commutator is running. The deflection increases with the speed of the commutator, and its magnitude can be greatly changed by rubbing the ebonite posts or the disk of the commutator. In magnitude the deflection may vary from zero to two or three centimeters and may be either positive or negative, depending upon the distribution of the charges. Rubbing the posts produces a deflection in one direction; rubbing the disk produces a deflection in the opposite direction. By passing a flame over the ebonite surfaces or by gently rubbing one or other of them

the deflection can usually be reduced to a small value. A capacity determination was never attempted if the deflection exceeded 5 millimeters, and for most of the work it was under 2 millimeters. This deflection corresponds to the transfer of a definite quantity of electricity and is independent of the actual capacity being measured. Hence, since the capacity desired is the difference of two measured capacities, this effect will not affect the determination of the capacity desired unless the charges on the commutator are different for the two measurements. Therefore if this effect remains constant, or varies uniformly, it will be eliminated along with the capacity of the leads and of the commutator. However, we can not count on it remaining constant, or varying uniformly for any great length of time, so it is advisable to eliminate it as much as possible from every measurement. Two methods have been used to effect this. The first method employed consisted in measuring a capacity, then reversing the battery and the galvanometer and immediately measuring the same capacity again; if the mean of this effect was the same for both measurements, the mean of the two will be the value of the capacity freed from this effect. In this method the zero of the galvanometer was the zero on open galvanometer circuit. The second method consisted in taking for galvanometer zero its zero on closed circuit, the battery circuit being open. This zero is a spurious one, depending, as we have seen, upon the effect under consideration; hence measurements thus made will be free from this effect provided it remained constant during the observations. When the effect was very variable, the runs were made longer and the zero of the galvanometer was reset at intervals during the run. The variation in this effect was the greatest and most annoying source of error met with in the electromagnetic measurements.

It is evident that the errors mentioned under the second and the third groups also produce a deflection of the galvanometer when the battery is disconnected, and are eliminated by the methods just outlined.

(5) The fifth source of error is of the same nature as the fourth except that the inducing charges are upon the metal portions of the brushes and are maintained by the battery. Though always present when a capacity is measured by means of such a commutator, it can never be independently observed and allowed for. It takes

its rise in the fact that the commutator segment has a somewhat different capacity when it breaks contact with one brush from what it has when it breaks contact with the other. The difference in capacity is, however, independent of the capacity connected with the commutator; hence when the capacity is determined by the method of differences, as is always done, the final result is unaffected by this source of error.

49. EFFECT OF VARIATIONS OF SPEED.

When considering the question of galvanometers, we arrived at the conclusion that even at the highest speed we ever used the duration of contact of the commutator brushes is ample to ensure completeness of the charge and of the discharge. Nevertheless it appeared desirable to test this point experimentally. For this purpose the difference between the capacity of the cylindrical condenser built up with three sections and no guard cylinder and the capacity of the same condenser when built up with but two sections was measured at different speeds, with the following results:

TABLE XVI.

Effect of speed.

Revolutions per minute	Charges per second	Capacity
970	259	76.570×10^{-6} mf
1,130	301	76.558 "
1,175	313	76.556 "
1,280	341	76.555 "
1,320	352	76.556 "
1,370	365	76.556 "
1,450	387	76.556 "
1,530	408	76.575 "
1,660	443	76.560 "
1,680	448	76.559 "
1,700	453	76.560 "
1,850	493	76.555 "
1,870	499	76.579 "
Mean		76.561 "

These results show that within this range the measured capacity is independent of the speed. When we consider that the capacity given above is the difference between two larger capacities, the greater of which was 232×10^{-6} mf the agreement is all that can be expected. The average variation from the mean is only 0.006×10^{-6} mf; that is, about one in 40,000 of the larger capacity, and there is no evidence that the value changes with the speed. The inner cylinder used in this work was not capped.

A few measurements of the capacity of ball B at different frequencies were made. These also indicate that the capacity as measured is independent of the frequency. The values found are given in Table XVII.

TABLE XVII.

Variations with speed. Ball B.

Revolutions per minute	Charges per second	Capacity
1,070	288	32.979×10^{-6} mf
1,250	333	32.981 "
1,350	360	32.986 "
1,600	427	32.977 "
1,800	480	32.983 "
	Mean	32.981 "

Here the average variation from the mean is only 0.0026×10^{-6} mf; that is, about one in 13,000 of the mean. When we remember that 0.0026×10^{-6} mf is the capacity of a sphere having a radius of but 23 μ, this variation is not astounding.

50. EFFECT OF VARIATION OF VOLTAGE.

The battery, usually of about 200 volts, for charging the condenser consisted of small storage cells. This high voltage was chosen in order to reduce the relative importance of the electrostatic effect of the commutator. To test the effect of different voltages, a key was so connected as to allow the voltage to be suddenly changed between certain limits. Even when these limits were so different as 32 and 214 volts, the change produced no observable change in the

TABLE XVIII.

Spherical Condenser, Ball B.

July 31, 1905. Mean air temperature 25°.8 C. Mean temperature of resistances 26°.0 C for all. Hygrometer 62%.

Run	Chr. time, 80,000 charges	Mean	a Nominal Resist.	K	K_1	K_2	k	$K-K_1=c_2$	K_2-k	$K-K_2=c_1$	K_1-k	Remarks
1	179.550	179.559	2162	.....	.....	48.392	.....	.....	.....	.....	.....	
2	179.568	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
3	179.617	179.600	2322	51.982	.....	.....	.....	1.672	.....	3.590	.....	
4	179.583	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
5	179.472	179.460	2249	.....	50.310	.....	.....	.....	.....	.....	.....	
6	179.449	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
7	179.700	179.719	613	.....	.....	.....	13.723	.....	.....	.....	36.582	
8	179.738	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
9	179.434	179.428	2249	.....	50.301	.....	.....	.....	.....	.....	.....	
10	179.422	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
11	179.556	179.564	2322	51.972	.....	.....	.....	1.671	.....	3.595	.....	
12	179.573	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
13	179.513	179.506	2162	.....	.....	48.377	.....	.....	34.662	.....	.....	
14	179.498	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
15	179.626	179.614	613	.....	.....	.....	13.715	.....	.....	.....	.....	After run 16, stopped the machine and adjusted its brushes and commutator brushes.
16	179.602	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
17	180.018	179.971	613	.....	.....	.....	13.742	.....	.....	.....	.....	
18	179.924	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
19	179.604	179.604	2162	.....	.....	48.404	.....	.....	34.662	.....	.....	
20	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	Very unsteady; run worthless; stopped for lunch.
21	179.474	179.461	2162	.....	.....	48.365	.....	.....	.....	.....	.....	
22	179.449	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	

23	179.517	179.509	2322	51.956				1.676		3.591		
24	179.501											
25	179.365	179.354	2249		50.280						36.580	
26	179.343											
27	179.450	179.421	613				13.700					Balance unsteady. After 28, stopped the machine and fixed the brushes.
28	179.392											
29	179.489	179.476	2249		50.314						36.593	
30	179.463											
31	179.720	179.692	613				13.721					
32	179.664											
33	179.532	179.512	2162			48.379			34.661			
34	179.493											
35	179.616	179.599	613				13.714					
36	179.582											
37	179.513	179.503	2162			48.376			34.664			
38	179.494											
39	179.560	179.555	613				13.710					
40	179.550											
41	179.490	179.488	2162			48.372			34.664			
42	179.486											
43	179.552	179.513	613				13.707					
44	179.474											
							Mean	1.673	34.663	3.592	36.584	
							Capacity of charging wire		1.673		3.592	
							Capacity of Ball B		32.990×10^{-6} mf		32.992×10^{-6} mf	
							Correction for resistance and time		-0.010_1 " "		-0.010_1 " "	
							Capacity of Ball B at 20°.0 C		32.979_9 " "		32.981_9 " "	

balance of the bridge. Also for the plate condenser a few complete runs were made at different voltages. These likewise failed to indicate that the measured capacity depended in the least upon the voltage employed. Thus, the means of two runs at different voltages gave for the capacity of the condenser, leads, and commutator the following values:

$$\text{At } 32 \text{ volts, } C = 137.481 \times 10^{-6} \text{ mf,}$$
$$\text{At } 214 \text{ volts, } C = 137.486 \times 10^{-6} \text{ mf.}$$

A difference of only 5 parts in 137,000; the sensibility, and hence the accuracy with which the speed can be controlled, is of course much less at the lower voltage than at the higher one.

51. METHODS FOR REDUCTION OF OBSERVATIONS.

Four different methods for reducing the observations were employed, depending upon the chronograph and the method employed in obtaining them.

In Table XVIII is shown the form of reduction employed in the earlier stages of the work. At this time we were using an electrically driven chronograph of a common form; the electrostatic effect of the commutator was eliminated by balancing to the galvanometer zero obtained from closed galvanometer but open battery circuit (see p. 567); after making a three-minute chronograph run the battery and galvanometer were both reversed and another run was made. The slight difference in the speed in the two cases is due to slight errors in the elimination of the electrostatic effect. The chronograph time for 80,000 charges is given in the second column of the table, and the mean of these times for each pair of runs taken as just described is given in the third column. This mean corrected to mean solar time (in this case a siderial time chronometer was used) and combined with the assumed values of the resistances, give the capacities recorded in columns 5–8. The values assumed for the resistances are those based upon the most recent preceding comparison of the coils with the standards. As stated in section 44, these assumed values were later corrected for the seasonal changes in the resistances. This correction is combined with the correction for the temperature of the condenser and applied to the final result as given at the foot of the table. In the fourth column of the table

is given the nominal value of the resistance in the arm *a* of the bridge (see Fig. 29); the resistances in the other arms remained unchanged throughout the day. In the fifth column is given the electromagnetic capacity, K, of the commutator, lead wires, and ball with both charging wires in position. Removing the top charging wire, w_2 (see Fig. 2) the electromagnetic capacity of the remaining portion is denoted by K_1 and is given in column 6. Replacing w_2 and removing the side charging wire, w_1, the capacity becomes K_2 as given in the seventh column. Finally, removing both charging wires we obtain the capacity of the commutator and leads as given under k in column 8. The effective capacity of the upper charging wire, c_2, is the difference between K and K_1, that of the side wire, c_1, is $K-K_2$. The capacity of the ball increased by the effective capacity of the upper charging wire is equal to K_2-k; the capacity of the ball increased by the capacity of the side charging wire is equal to K_1-k (see p. 444). These differences are given in the last four columns of the table. The unit in which all these capacities are expressed is the one-millionth of a microfarad.

In Table XIX is shown the method employed in reducing the results obtained by means of the differential galvanometer and the new form of chronograph.

Having decided upon a speed of 1,500 revolutions per minute (400 charges per second) as suitable for the work, a table was constructed giving the values of the capacity corresponding to given values of the resistance *a*. In this table (not given in this paper) the values of *a* (in the region with which we are concerned) differed by steps of one ohm each, so that it was easy to interpolate to a hundredth of an ohm in *a*. These capacities are of course based upon assumed values for the resistances of *c*, *d*, and the galvanometer, and so are but approximately correct. From this table we get the values given in the third column of Table XIX. In the fourth column are given the corrections k_1 (in parts per 1,000), which must be added to the approximate capacity on account of the speed of the machine being slightly different from that corresponding to exactly 1,500 revolutions per minute. As explained on page 564, k_1 is merely twice the tangent of the angle of deviation of the rows of second dots on the chronograph sheet from the generating lines of the drum. In the fifth column is the sum of k_1 and k_2, where k_2 is the correction

TABLE XIX.

Spherical Condenser, Ball B.—Differential Method.

November 24, 1905. Mean air temperature 21°.0 C. Mean temperature of resistances 20°.0 C for all. Hygrometer 50%.

Run	a corrected	Approx. capacity	k_1	k_1+k_2	Total correct.	k	K_2	K_1	K	$K-K_1$	K_2-k	Mean	Remarks
1	2119.61	51.691	+0.38	+0.33	+17	.....	.....	.....	51.708	.....	.....	.....	Speed poor.
2	2051.61	50.069	+0.25	+0.20	+10	.....	.....	50.079	.....	.....	.....		
3	2120.61	51.714	−0.56	−0.61	−32	.....	.....	.....	51.682	1.604	.....		
4	2052.61	50.093	−0.28	−0.33	−16	.....	.....	50.077	.....	1.608	.....	1.610	
5	2119.61	51.691	−0.01	−0.06	− 3	.....	.....	.....	51.688	1.612	.....		
6	2052.61	50.093	−0.34	−0.39	−19	.....	.....	50.074	.....	1.614	.....		
7	2119.61	51.691	−0.01	−0.06	− 3	.....	.....	.....	51.688	.....	.....		
8	546.51	13.557	+1.98	+1.93	+26	13.583	.....	.....	.....	.....	.....		
9	547.51	13.581	−0.17	−0.22	− 3	13.578	.....	.....	.....	.....	.....		
10	1971.16	48.147	+0.79	+0.74	+36	.....	48.183	.....	.....	.....	34.606		
11	547.51	13.581	−0.37	−0.42	− 6	13.575	.....	.....	.....	.....	.....		
12	1972.16	48.171	−0.19	−0.24	−12	.....	48.159	.....	.....	.....	34.584		
13	547.51	13.581	−0.39	−0.44	− 6	13.575	.....	.....	.....	.....	.....		
14	1972.16	48.171	−0.13	−0.18	− 9	.....	48.162	.....	.....	.....	34.586		
15	547.51	13.581	−0.22	−0.27	− 4	13.577	.....	.....	.....	.....	.....	.....	After 15, stopped for lunch.
16	547.51	13.581	−0.34	−0.39	− 5	13.576	.....	.....	.....	.....	.....	34.590	
17	1972.16	48.171	−0.19	−0.24	−12	.....	48.159	.....	.....	.....	.....	.....	Interchanged coils.
18	1972.16	48.171	−0.26	−0.27	−13	.....	48.158	.....	.....	.....	34.586		
19	547.51	13.581	−0.80	−0.81	−11	13.570	.....	.....	.....	.....	.....	.....	Interchanged coils.

20	547.51	13.581	−0.52	−0.57	− 8	13.573	.....	.....	.....	.....	.....	.....	
21	1972.16	48.171	−0.02	−0.07	− 3	.....	48.168	.....	.....	.....	.....	.....	Interchanged coils.
22	1972.16	48.171	−0.18	−0.19	− 9	.....	48.162	.....	.....	.....	34.591		
23	547.51	13.581	−0.44	−0.45	− 6	13.575	.....	.....	.....	.....	34.586		
24	1972.16	48.171	−0.19	−0.20	−10	.....	48.161	.....	.....	.....	.....	.....	After this, discharge.
25	1972.16	48.171	−0.24	−0.25	−12	.....	48.159	.....	.....	.....	34.587		
26	547.51	13.581	−0.84	−0.85	−12	13.569	.....	.....	.....	.....	34.588		
27	1972.16	48.171	−0.33	−0.34	−16	.....	48.155	.....	.....	.....	.....	34.588	Interchanged coils.
28	1972.16	48.171	−0.30	−0.35	−17	.....	48.154	.....	.....	.....	.....		
29	547.51	13.581	−1.08	−1.13	−15	13.566	.....	.....	.....	.....	34.589		
30	1972.16	48.171	−0.27	−0.32	−15	.....	48.156	.....	.....	.....	.....		

	Charge	Discharge
∴ Capacity of ball and wire	34.590×10^{-6} mf	34.588×10^{-6} mf
" " wire	1.610 " "	1.610 " "
" " Ball B	32.980 " "	32.978 " "
Corrections for temperature and errors in the assumed values of the resistances	-0.002_3	-0.002_3
Capacity of Ball B at 20°.0 C	$32.977_7\times10^{-6}$ mf	$32.975_7\times10^{-6}$ mf

TABLE XX.

Spherical Condenser, Ball B.—Speed controlled by auxiliary bridge.

Mean air temp., 21°.0 C. Mean temp. of resistances, 21°.0 C.

November 12, 1906.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Capacity Measured	a	$c'-49{,}500$ Settings				Mean	$c'_0-49{,}500$	$c'-c'_0$	Corrections in parts per 1,000				Correction in Ω	Correction to a	Capacity	Capacity of wire	Capacity ball and wire
									s	cd	Ap	Total					
Chronograph run							76.6										
Wire in	2137.5	99.0	99.0	93.0													
Bridge.		99.0	97.0	90.0		96.2	76.6	+19.6	+0.39	−1.00	−0.04	−0.65	−1.39	+0.19	53.4075	1.6143	
Wire out	2073.2	88.0	90.0														
		89.0	87.0			88.5		+11.9	+0.24	−1.00	−0.04	−0.80	−1.66	+0.19	51.7932		
Wire in	2137.5	90.0	85.0														
Bridge.		88.0	83.0			86.5	76.7	+ 9.8	+0.20	−1.00	−0.04	−0.84	−1.80	+0.19	53.3972	1.6142	
Wire out	2073.2	80.0	77.0														
		79.0				78.7		+ 2.0	+0.04	−1 00	−0.04	−1.00	−2.07	+0.19	51.7830		
Wire in	2137.5		89.0	86.0													
Bridge.			87.0	88.0		87.5	76.8	+10.7	+0.22	−1.00	−0.04	−0.82	−1.75	+0.19	53.3985	1.6103	
Wire out	2073.2	86.0	85.0	83.0													
		81.0	83.0			83.6		+ 6.8	+0.14	−1.00	−0.04	−0.90	−1.86	+0.19	51.7882		
Wire in	2137.5	90.0	87.0	89.0	85.0												
Bridge.		87.0	87.0	90.0	84.0	87.4	76.9	+10.5	+0.21	−1.00	−0.04	−0.83	−1.77	+0.19	53.3980	1.6125	
Wire out	2073.2	82.0	83.0	84.0													
		80.0	79.0	80.0		81.3		+ 4.4	+0.09	−1.00	−0.04	−0.95	−1.97	+0.19	51.7855		
Chronograph run							77.0										
Ball and commutator	1998.5	82.0	82.0	81.6													
Bridge.			82.0	80.0		81.5	77.4	+ 4.1	+0.08	−1.00	−0.04	−0.96	−1.92	+0.19	49.9192		
Commutator	614.0	80.0	78.0														34.5820
		76.0	81.0			78.7		+ 1.3	+0.03	−1.00	−0.01	−0.98	−0.60	+0.09	15.3372		

Ball and commutator	1998.5	81.0	78.0												
Bridge.		78.8	77.0	.....	78.5	78.1	+ 0.4	+0.01	−1.00	−0.04	−1.03	−2.06	+0.19	49.9158	
Commutator	614.0	71.0													34.5808
		73.0	.....	.....	72.0	.....	− 6.1	−0.12	−1.00	−0.01	−1.13	−0.69	+0.09	15.3350	
Chronograph run		.....	.....	.....		78.5									
Ball and commutator	2045.2	73.0	79.0	76.0	70.0										
Charge.		74.0	72.0	76.0	68.0										
Commutator	618.5	66.0	87.0	77.0											
		71.0	83.0	79.0	.....	77.2									
Chronograph run		.....	.....	.....	.....	81.0									
Ball and commutator	1998.2	86.5	86.6												
Bridge S.		85.3	84.5	.....	.....	85.7	81.0	+ 4.7	+0.09	−1.00	−0.06	+0.97	−1.94	+0.19	49.9112
Commutator	613.7	79.0													34.5812
		78.0	.....	.....	.....	78.5	.....	− 2.5	−0.05	−1.00	−0.01	−0.96	−0.59	+0.09	15.3300

Ball and commutator	1998.5	81.0	78.0												
Bridge.		78.8	77.0	.....	78.5	78.1	+ 0.4	+0.01	−1.00	−0.04	−1.03	−2.06	+0.19	49.9158	
Commutator	614.0	71.0													34.5808
		73.0	.....	.....	72.0	.....	− 6.1	−0.12	−1.00	−0.01	−1.13	−0.69	+0.09	15.3350	
Chronograph run		.....	.....	.....		78.5									
Ball and commutator	2045.2	73.0	79.0	76.0	70.0										
Charge.		74.0	72.0	76.0	68.0										
Commutator	618.5	66.0	87.0	77.0											
		71.0	83.0	79.0	.....	77.2									
Chronograph run		.....	.....	.....	.....	81.0									
Ball and commutator	1998.2	86.5	86.6												
Bridge S.		85.3	84.5	.....	.....	85.7	81.0	+ 4.7	+0.09	−1.00	−0.06	+0.97	−1.94	+0.19	49.9112
Commutator	613.7	79.0													34.5812
		78.0	.....	.....	.....	78.5	.....	− 2.5	−0.05	−1.00	−0.01	−0.96	−0.59	+0.09	15.3300

Mean	1.6128	34.5821
Capacity of wire		1.6128
Capacity ball B		32.9693
Correction for temp. and resist.		−0.0023
Capacity of ball B at 20·0 C.		32.9670

(in parts per 1,000) to be added to the capacity on account of the differences between the true values of the resistances of c, d, and the galvanometer, and the values assumed in the construction of the table from which we obtain the approximate capacity as given in column 3. In the sixth column is given the correction $k_1 + k_2$ expressed in thousandths of a micro-microfarad. The symbols at the heads of the following columns have the same meaning as in the former case.

As before, the values of the resistances assumed to be correct when the observations are reduced are slightly in error owing to the seasonal variations in the resistances. The correction for this error and the correction for the temperature of the condenser differing from 20° C are combined and applied to the final result, as shown at the bottom of the table.

The charging current was measured in the first 24 runs, and the discharging current was measured in the rest. These have been separately averaged and reduced, and give results differing by but 1 part in 16,000.

In order to test the equality of the coils of the galvanometer under working conditions four pairs of runs (17–18, 19–20, 21–22, 27–28) were taken, the second in each pair being taken immediately after the first, but with the galvanometer coils interchanged. No certain difference is observable.

As before, the electrostatic effect was eliminated by balancing to the displaced zero of the galvanometer.

In Table XX is shown the form used in the reduction of observations obtained by the method (page 558) in which the speed is controlled by means of an auxiliary bridge containing a large capacity. During this series four chronograph runs were taken, as shown in the table; the temperature of the large condenser was not kept perfectly constant so that the resistance c'_0 required to balance the auxiliary bridge when the speed of the machine was exactly 1,500 revolutions per minute gradually increased.

The first part of the table is taken up with the determination of the electromagnetic capacity of the upper charging wire (w_2). "Wire in" denotes that this wire is in place so as to touch the ball and to dip into the mercury cup that serves to connect it with the commutator; "Wire out" denotes that it has been entirely removed. Throughout this portion of the work the side charging wire w_1 was continuously in position so as to connect the ball to the commutator.

In column 2 is given the nominal value of the resistance in arm a of the main bridge; in the next four columns are given the values of the arm c' (decreased by 49,500 ohms) of the auxiliary bridge which correspond to a simultaneous balance of both bridges. The order in which these observations were taken is best illustrated by an example. Let us take the first set given in the table. The wire w_2, being in place, a is made equal to 2,137.5 ohms, and a simultaneous balance is found for $c' = 49,599.0$ ohms; the auxiliary observer now changes the portion of the c' arm that is near him and the main observer (not knowing the amount of this change) proceeds to obtain another simultaneous balance of the bridges; this again occurs for $c' = 49,599.0$ ohms. The wire is now taken out, a is made equal to 2,073.2 ohms, and two simultaneous balances are obtained, as before, for $c' = 49,588.0$ and 49,589.0 ohms. Then the wire is replaced, a changed to its former value, and balances are obtained for $c' = 49,599.0$ and 49,597.0 ohms, etc. The mean of these settings for each case is given in column 7. In the eighth column are given the values of $c'_0 - 49,500$ that correspond to the balance of the auxiliary bridge for a speed of exactly 1,500 revolutions per minute; these are obtained by interpolating between the four chronograph runs taken during the day. The following column contains the difference between c' and c'_0, which (divided by c'_0) gives the correction s (column 10) in parts per 1,000 to be added to a in order to obtain a balance for the even speed of 1,500 revolutions per minute. Column 11 contains the further correction to be added to a in order to compensate for the fact that cd has a value different from that assumed in the calculation of the capacity for a given value of a. The next column (headed Ap) contains the correction that must be applied to the value of the capacity as calculated by the approximate formula $\left(C = \frac{a}{ncd}\right)$ for the bridge in order to obtain the true capacity; the third and the fourth numbers from the end of this column are the corrections that must be applied on account of the galvanometer resistance being different from that assumed in the construction of the table giving the capacity corresponding to a given value of a when the differential method is used. The sum of these three corrections is given in column 13; the product of these sums by

one-thousandth of the corresponding a is the correction in ohms to be applied to a; this is given in column 14. In column 15 is given the correction to be added to a on account of the errors in the nominal values of the coils of which it is composed. The value of a as given in column 2, increased by the corrections in columns 14 and 15, is exactly 40 times the true electromagnetic capacity, expressed in micro-microfarads, if the bridge method is used. If the differential method is used, the true capacity is that given by our table for this corrected value of a. These capacities are given in column 16, and their differences (corresponding to the capacity of the wire or of the ball and wire) are given in columns 17 and 18.

The runs marked "Bridge" in the first column were obtained by the bridge method by the use of the Weston (low resistance) galvanometer; that marked "Charge" was by the differential method, the charging current being measured; and the last, marked "Bridge S," is by the bridge method, the Sullivan (high resistance) galvanometer being used.

The three methods substantially agree, and give 32.967×10^{-6} mf as the capacity of ball B at 20°o C.

52. RESULTS BY BALL A.

Many preliminary determinations of the electromagnetic capacity of ball A were made, but all of those made before March 7, 1905, are unreliable on account of errors in the determination of the capacity of the charging wire, heating of the resistance in arm d, leakage, or other cause. On March 6, 1905, the 100,000 ohms resistance composing arm d was placed in oil. By this time the method for determining the capacity of the charging wire had been perfected, and the leakage had been reduced so as to be usually negligible. This date is therefore taken as the beginning of the actual measurements. However, there would occasionally be enough leakage over the ebonite bushing, which served as a guide for the side charging wire, to render the determination of the capacity of the charging wires and of the ball, when measured by means of the side charging wire, uncertain. This condition of affairs continued until April 4, 1905, when the bushing was replaced by one extending further beyond the surfaces of the shell and having the hole enlarged at each end so as to increase the leakage path. After this no further trouble from this cause was experienced.

Owing to this trouble some of the observations taken between March 7 and April 4 had to be discarded. *All* the determinations of the capacity of ball A, made after April 4, 1905 (except 6 single determinations out of a total of 65), are given in Table XXI. The observations omitted compose two groups of consecutive observations and every indication pointed to the fact that during these observations the commutator brushes were too loose.

TABLE XXI.

Results by Ball A.

Date		Wire	C	δ_1	A	δ_2	N	W	v	Mean
1905										
Mar.	7	1.516	13.928	3	55.920	3	3	½	2.9965	2.99660
	7	3.339			55.915	8	2	½	2.9967	
Apr.	10	1.454	14.026	3	55.922	2	3	3	2.9965	
	10	3.448			55.919		1	1	2.9966	
	17	1.454	13.883	2	55.929	1	2	2	2.9963	
	26	1.495	14.106	26	55.928	9	3	2	2.9963	
	26	3.484			55.932		1	1	2.9962	2.99614
May	1	1.482	14.031	2	55.944	3	4	3	2.9959	
	1	3.433			55.944	4	4	3	2.9959	
	2	1.475	14.014	15	55.939	5	5	4	2.9960	
	2	3.429			55.939	6	3	2	2.9960	
July	14	1.541	13.787	9	55.923	1	3	3	2.9964	
	15	1.504	13.741	15	55.962	2	2			
	15	3.476			55.929	4	2	1	2.9963	
	18	1.512	13.761	20	55.950	2	3	2	2.9959	2.99620
	18	3.471			55.934	12	2	1	2.9963	
	19	1.505	13.794	2	55.942	2	2	1	2.9961	
	19	3.472			55.944	2	2	1	2.9961	
	21	1.574	13.749	3	55.928	1	2	2	2.9963	
	21	3.466			55.928	2	2	2	2.9963	
	22	1.572	13.766	1	55.938	4	2	1	2.9960	
	22	3.472			55.934	6	2	1	2.9962	2.99629
	24	1.575	13.758	5	55.932	1	2	2	2.9962	
	24	3.474			55.932	2	2	2	2.9962	
	25		13.692	3	55.926	4	4	4	2.9964	
	25				55.920		1	1	2.9965	

The only other result with this ball that does not appear in the final mean is the first given under date of July 15, 1905, in the table. Though everything appeared to be going nicely when these observations were taken, except that the humidity was high (70%), the results when reduced were found to be very erratic.

All the determinations with ball A were made by the bridge method.

The second column of the table contains the capacity of the charging wire; the smaller capacity (1.5) is that of the wire which passed through the hole in the top of the shell, the other (3.5) is that of the one which passed through the hole at the equator of the shell. In the third column under C is given the capacity of the commutator and leads; in the next column is given δ_1, which is the average variation of the capacity of the commutator and leads from the mean value given under C. In the fifth column under A is given the capacity of ball A at 20°.0 C (all corrections applied), and in the following column is given the average variation δ_2 of its measured capacity from this mean. N is the number of distinct determinations included in the mean value of the capacity given under A, and W is the weight assigned to this mean. In the ninth column under v is the square root of the ratio of the electrostatic capacity of ball A (corrected for the holes in the shell, ebonite bushings, etc.) to the value given in column A.

The unit in which all the electromagnetic capacities are expressed is the one-millionth of a microfarad. The average variations from the mean (δ_1 and δ_2) are expressed in terms of units in the last place—that is, in terms of one-thousandth of a micro-microfarad.

It will be noticed that δ_1 is as a rule considerably greater than δ_2. The cause of this is obvious. Every change in the adjustment of the brushes changes the value of the commutator capacity, and the brushes were usually readjusted several times in the course of a day. This sometimes caused the commutator capacity to vary over wide limits, and so made δ_1 large. But the value of the capacity of the ball is the difference between the measured capacity of the ball, commutator, and leads, and the mean of two measurements of the capacity of the commutator and leads, one being taken before and the other after the measurement with the ball. The commutator brushes are never touched during such a set of three measurements,

so this difference is independent of the large changes in the commutator capacity and is almost independent of the small ones.

The results naturally fall into four groups, which have been separately averaged. After each group the condenser was entirely dismantled and was reassembled again just before the beginning of the results recorded in the following group.

The mean value of δ_2 is but 0.0037 micro-microfarads, which shows that on the average a single measurement of the capacity of ball A departed from the mean of all measurements of the same day by no more than the capacity of a sphere of radius equal to 33 microns, or 7 parts in 100,000.

The weighted mean of these 64 independent determinations is

$$v = 2.99621 \times 10^{10}$$

The average variations of the results in the ninth column of the table from this mean is 0.00018×10^{10}, or 6 parts in 100,000.

53. RESULTS BY BALL B.

The preliminary work having been completed with ball A, ball B was adjusted in the shell on March 13, 1905, and the first determination of its electromagnetic capacity was made on the following day. As stated in the preceding section, we were occasionally troubled with leakage until April 4, 1905, so that some of the results in March had to be rejected. All the determinations of the capacity of ball B made after the latter date (except 3 single determinations out of a total of 202) are given in Table XXII. These omitted determinations were the last observations taken on November 21, 1905, when the commutator brushes were badly worn, and the commutator was working badly. There were in addition a few preliminary runs taken with the differential galvanometer. These were taken for the purpose of studying the instrument and the method and were not used for the purpose of obtaining a measurement of the capacity of the condenser.

Table XXII is analogous to Table XXI; the only change is that the column containing the capacity of the ball is headed B, and a column is inserted between this column and the column containing the mean deviations (δ_2). This column, which is marked M, con-

tains the designations of the methods employed; b, signifies that the Maxwell bridge method has been used; d, that the *discharge* has been measured by the differential galvanometer method; and c, that the *charge* has been measured by the same method. The results obtained on September 1 and 6, 1905, at different speeds have been averaged and recorded separately; the results for each day are arranged in the order of increasing speed. The measurements in October and November of 1906 were made by the method in which

TABLE XXII.

Results by Ball B.

Date		Wire	C	δ_1	B	M	δ_2	N	W	v	Mean
1905											
Mar.	14	1.601	13.935	2	32.984	b	0	3	3	2.9962	
	14	3.464			32.980	b	2	3	3	2.9964	
	16	1.603	13,932	4	32.984	b		1	1	2.9962	
	18	1.609	13.959	1	32.984	b		1	1	2.9962	
	21	1.616	13.996		32.977	b		1	1	2.9965	
	22	1.627	13.965	3	32.980	b	2	5	5	2.9964	2.99633
	22	3.449			32.977	b	4	3	3	2.9965	
	24	1.621	13.975	4	32.984	b	2	3	3	2.9962	
	24	3.440			32.983	b	2	3	3	2.9962	
	25	1.632	13.962	2	32.981	b	2	3	3	2.9963	
	25	1.632	13.984	1	32.980	b	1	3	3	2.9964	
July	26	1.656	13.675	1	32.978	b	1	2	2	2.9965	
	26	3.584			32.979	b	2	2	2	2.9964	
	27	1.662	13.683	5	32.982	b	8	2	2	2.9963	
	27	3.582			32.985	b	2	2	2	2.9962	
	28	1.666	13.691	7	32.976	b	1	4	3	2.9965	2.99635
	28	3.587			32.981	b	4	4	4	2.9963	
	31	1.673	13.716	9	32.980	b	1	3	3	2.9964	
	31	3.592			32.982	b	5	5	5	2.9963	
Sept.	1	1.672	13.814	11	32.977	b	3	5	5	2.9965	
	1		13.797	1	32.979	b	3	2	2	2.9964	
	1		13.756	1	32.974	b	3	2	2	2.9967	2.99644
	6	1.666	13.786	1	32.984	b	2	2	2	2.9962	
	6		13.764	8	32.976	b	4	4	4	2.9965	
	6		13.753	1	32.981	b	1	2	2	2.9962	

TABLE XXII—Continued.

Results by Ball B—Continued.

Date		Wire	C	δ_1	B	M	δ_2	N	W	v	Mean
1905.											
Oct.	30	1.623	13.507	1	32.985	d	3	2	2	2.9962	
	31	1.645	13.532	0	32.975	b	3	2	2	2.9966	
	31		13.541	1	32.980	d	1	3	3	2.9964	
	31		13.557	4	32.979	c	1	3	3	2.9964	
Nov.	2		13.460	8	32.982	b	2	4	4	2.9963	
	3	1.625	13.480	8	32.983	b	5	3	2	2.9962	
	3		13.474	7	32.980	c	1	3	3	2.9964	
	3		13.528	4	32.980	d	4	3	3	2.9964	
	6	1.635	13.549	3	32.974	d	3	4	4	2.9966	
	6		13.548	2	32.979	c	6	4	4	2.9964	
	6		13.543	3	32.975	b	3	4	4	2.9966	
	9	1.632	13.452		32.980	d		1	1	2.9963	2.99637
	10	1.625	13.429	4	32.976	d	1	2	1	2.9965	
	18	1.615	13.721		32.982	d		1	1	2.9963	
	20	1.618	13.569	4	32.982	d	1	3	3	2.9963	
	20		13.564	3	32.982	c	3	3	3	2.9963	
	23	1.611	13.544	3	32.986	b	5	4	4	2.9961	
	23		13.550	3	32.998	d	6	2	2	2.9956	
	23		13.545	4	32.980	c	6	8	7	2.9964	
	24	1.610	13.576	2	32.978	c	6	4	4	2.9964	
	24		13.568	2	32.976	d	2	3	3	2.9965	
	25	1.607	13.600	1	32.979	d	1	2	2	2.9964	
	25		13.632	22	32.979	c	4	4	4	2.9964	
	29	1.619	13.702	11	32.977	b	2	5	5	2.9965	
1906.											
Oct.	30	1.605	15.393	1	32.975	c	0	2	2	2.9966	
	30		15.410	2	32.977	c	1	8	8	2.9965	
	30		15.399	1	32.978	c	0	3	3	2.9965	
	30		15.415	1	32.974	c	0	2	2	2.9966	
Nov.	1	1.604	15.411	2	32.971	c	2	2	2	2.9967	
	1		15.407	2	32.974	c	1	10	10	2.9966	
	12	1.613	15.337	1	32.967	b	0	5	5	2.9969	
	12		15.335	0	32.966	b	1	4	4	2.9970	2.99659
	12		15.333	2	32.970	c	2	8	8	2.9968	
	12		15.328	0	32.966	b	1	4	4	2.9969	
	15	1.609	15.154	0	32.981	c	2	6	6	2.9963	
	16	1.607	15.381	1	32.977	b	1	8	8	2.9965	
	16		15.381	1	32.980	c	1	6	6	2.9964	
	16		15.385	1	32.977	c	2	6	6	2.9965	
	16		15.390	2	32.980	c	1	7	7	2.9964	

the auxiliary bridge is employed (see p. 558). It will be noticed that δ_1 is here very much smaller than it is in the first part of the table; this is a consequence of the greater rapidity with which determinations can be made by this method; the commutator brushes are not touched at all during the determinations composing any one series.

These results fall into five groups, which have been separately averaged. As with ball A the condenser was entirely dismantled after each group and was reassembled again just before the beginning of the preceding group.

The mean value of δ_2 for the observations of 1905 is but 0.0030 micro-microfarads, or the capacity of a sphere of a radius of 27 microns. The mean value of **v** for this year is 2.99637. In October and November, 1906, when the other method was employed, the mean value of δ_2 was but 0.0011 micro-microfarads, or the capacity of a sphere of a radius of 10 microns. The mean value of **v** for this period is 2.99661. At the time these measurements were made we were aware that they departed from the mean, and every endeavor was made to locate the cause of the discrepancy. The resistances were carefully compared with the standards and also intercompared, all leakage paths were carefully tested, and on November 12 three methods were employed in determining the capacity. The first two results given under that date were determined by the bridge method, using the low-resistance galvanometer; the fourth was by the same method, but with the high resistance galvanometer, and the third was by the differential method, the charge being measured. The results by these three methods agree almost perfectly. The next day the guard cylinder condenser was assembled and one determination of the capacity was made. This value gave a remarkably low value for **v**. The next day several results were obtained that differed but little from the mean. The following days the results obtained by the cylinders gave rather low values for **v**, while those by the ball were rather high, but much lower than the values found on November 12. The cause of these high values has not yet been found. It may possibly be due to some abnormality in the dielectric constant of air during this period.

The weighted mean of the 147 independent determinations made in 1905 is

$$\mathbf{v} = 2.99637 \times 10^{10}$$

The average deviation of the results in the tenth column of the table from this value is

$$0.00011 \times 10^{10}, \text{ or four parts in } 100{,}000.$$

The weighted mean of the 81 independent determinations made in 1906 is

$$V = 2.99659 \times 10^{10}$$

The average deviation determined as before is

$$0.00016 \times 10^{10}, \text{ or five parts in } 100{,}000.$$

From these results we can get a good comparison of the results obtained by the three methods designated as b, c, and d; collecting the results obtained by different methods on the same day we obtain the data presented in Table XXIII.

TABLE XXIII.

Comparison of Different Methods.

Method.	b	c	d
1905.			
Oct. 31	2.9966	2.9964	2.9964
Nov. 3	2.9962	2.9964	2.9964
6	2.9966	2.9964	2.9966
23	*2.9961	*2.9964	*2.9956
Mean	2.99647	2.99640	2.99647
Nov. 20	.....	2.9963	2.9963
24	.....	2.9965	2.9964
25	.....	2.9964	2.9964
Mean	.....	2.99640	2.99637
1906.			
Nov. 12	2.9969	2.9968	.....
16	2.9965	2.9964	.....
Mean	2.99670	2.99660	.....

* Omitted from mean, balance hard to maintain, brushes bad.

54. RESULTS BY CYLINDERS WITHOUT GUARDS.

A series of measurements were made with the upper ends of both cylinders, and with the lower end of the outer one open. These are assembled in Table XXIV. The numbers at the tops of the columns indicate the two capacities of which the difference is given in that column; for example, 1. 2. 3–1. 2 at the head of the second column indicates that each number contained in this column is the

TABLE XXIV.

Difference in the Capacity produced by Changing the Height.

Date.	1.2.3−1.2	W	1.2.3−1.3	W	1.3.2−1.3	W	1.3.2−1.2	W	1.2−1	W	1.3−1	W
1905												
Mar. 24	76.542	1							76.152	1		
Apr 18									76.245	2		
19			75.683(?)	1								
20			76.137	1	76.120	1			76.182	1	76.630	1
22					76.134	1	76.546	2	76.147	2	76.565	4
May 2			76.172	1					76.204	2		
3	76.589	2			76.143	2	76.606	2	76.195	2		
8	76.582	1			76.121	1			76.185	2	76.605	1
9	76.558	2	76.078	2			76.590	2	17.178	3		
Aug. 1	76.564	6			76.132	4			76.174	5	76.618	4
2	76.553	4	76.133	4	76.129	4			76.175	4	76.627	4
3	76.553	3										
	76.577	3										
	76.558	2										
23	76.565	2										
	76.555	2										
24	76.580	3										
	76.578	2										
	76.573	1										
25	76.564	2										
	76.523	1										
	76.519	1										
31	76.554	4										
	76.551	4										
	76.554	2										
Sept. 7	76.563	4										
	76.562	3										
	76.546	3										
12	76.552	3										
13	76.558	3										
Mean	76.560		76.125		76.131		76.581		76.183		76.605	
Correction	−0.005		−0.005		−0.005		−0.005		−0.005		−0.005	
	76.555		76.120		76.126		76.576		76.178		76.600	

difference between the capacity of the condenser as composed of sections 1, 2, and 3 and its capacity as composed of sections 1 and 2. As before, the unit of capacity employed is one micro-microfarad.

While the individual values are rather variable, the means are probably fairly accurate and show that the first section added invariably produces a greater increase in the capacity of the system than the same cylinder produces when it is the second section added. This is exactly what we have found should be observed on account of the distribution of charge on the outside of the outer cylinder. It was not due to the change in position of the upper end with respect to objects above it, for the upper end never came closer than 80 cm to any of these objects, while by actual trial it was found that a metal plate could be brought down from above to within about 10 cm of the upper end of the condenser before it appreciably affected the balance of the bridge.

Taking the weighted means from Table XXIV we find the following values for the increase in the capacity produced by the addition of one section:

TABLE XXV.

Section.	2	3
Section added first	76.178	76.600
" " second	76.124	76.557
Difference	0.054	0.043

Assuming 2.9963×10^{10} as the true value of **v**, the electromagnetic capacities of these sections are 76.085 and 76.518 micro-microfarads, respectively. Hence the measured capacity is too great by 124×10^{-5} and 107×10^{-5} of the total for the section first added, and 51×10^{-5} of the total for the section next added. Hence the section first added has a capacity higher than that calculated by 12 parts in 10,000, and the next section is higher by 5 parts in 10,000. Hence in order to use this method the shortest section of the cylinder should be at least 70 or 80 cm in length. This, however, will in turn necessitate a great increase in the accuracy of the electromagnetic measurements, for the determination will then depend upon the dif-

ference of two large quantities. For this reason it is clearly not desirable to use this method with open cylinders.

Capped cylinders.—Recently a few measurements were made on two days with both cylinders capped at the top. As stated on page 522, it seems that with the arrangement adopted no error is introduced by the fact that the lower end of the outer cylinder is not capped. The results are as follows:

TABLE XXVI.

Section.	2	3
First section added	76.109	76.531
Second " "	76.085	76.510
Difference	0.024	0.021

Here we still have a difference in the same direction as before, but a little less than half as large. The values of the capacities are much nearer their true values than those obtained when the ends were open, but for the first section added they are still too large. The capacities for the sections added second give us $v = 2.9963 \times 10^{10}$ for section 2, and 2.9964×10^{10} for section 3.

The only explanation of the high values found for the section added first seems to be that the capacity of the lower end of the cylinder, or of the leads and commutator, is appreciably affected by changing the height of the cylinder in spite of the tests described on page 522.

The method of procedure followed in this part of the work necessitates numerous buildings up of the condenser, and the results obtained serve as an indication of the reproducibility of this process. From the table we see that the average variation from the mean for any section is only 0.010 micro-microfarads. This is very good when we consider that this includes the variations in the commutator capacity, and these may well be large, as it requires some time to add and adjust (or to remove) a section.

55. RESULTS BY CYLINDERS WITH GUARDS.

Having found by experience that it is impossible to obtain consistent results with uncapped cylinders without guard cylinders,

the guard cylinders were constructed and after a few preliminary experiments the determination of v by the use of the guard cylinder condenser was begun on March 8, 1906. The use of the second set of commutator brushes (for charging the guard cylinders) necessitated the setting of all the brushes at rather unfavorable angles so that the vibration of these brushes was increased. This in turn made the brushes work loose more rapidly, and rendered the capacity of the commutator more variable. Because of this some of the measurements between March 8 and March 16 had to be discarded, but after the latter date the adjustment of the brushes was maintained by frequently testing with a milammeter, as already described. As a consequence, all observations after March 16, 1906, are included in Table XXVII, except 7 single determinations (out of a total of 457) when there was a slight leakage across the mica insulation separating the segments of the ring on which the side brush bears, causing the capacity of the commutator to be uncertain. On January 3 and 4, 1907, the balance of the bridge could not be maintained, so that the observations made were discarded; they indicated a variable leakage, probably between the segments of the commutator.

The construction of Table XXVII is similar to that of Table XXI, except that in this case there is no charging wire of which the capacity has to be determined. All these determinations were made by the differential method, except those of October 8 and 9, 1906, which were by the bridge method. All the results after October 9 were obtained by the method involving the use of the auxiliary bridge (page 558).

The condenser was dismantled and reassembled again after each group that has been separately averaged.

The average value of δ_2 for the observations taken before November 14, 1906, is 0.0150 micro-microfarads; this is only 1 in 10,000 of the capacity of the condenser, but is absolutely about 5 times as large as in the case of the spheres. This increase is due to the additional difficulties introduced by the second set of commutator brushes. The weighted mean of these 174 independent determinations is

$$v = 2.99626 \times 10^1$$

TABLE XXVII.

Results by Guard Cylinder Condenser.

Date		C	δ_1	Cylinder	δ_2	N	W	v	Mean
1906									
Mar.	8	13.470	2	152.814	7	5	3	2.9964	
	8	13.483	2	152.841	3	6	3	2.9962	
	9	13.453	38	152.818	7	5	2	2.9964	
	9	13.477	5	152.891	5	2	1	2.9957	
	10	13.520	16	152.799	8	8	4	2.9966	
	10	13.449	50	152.773	76	6	3	2.9968	2.99624
	12	13.665	125	152.860	17	6	2	2.9960	
	14	13.488	3	152.881	6	4	2	2.9958	
	15	13.394	15	152.851	6	4	2	2.9961	
	16	13.937	119	152.839	8	8	4	2.9962	
	20	14.107	92	152.883	28	13	3	2.9958	
	24	13.832	68	152.779	7	10	5	2.9968	
	26	13.942	21	152.796	21	9	2	2.9966	
	26	.....	.....	152.838	7	6	3	2.9962	2.99649
	28	13.905	49	152.825	4	4	2	2.9963	
	29	14.408	179	152.833	11	14	5	2.9962	
	30	14.902	29	152.782	18	12	4	2.9967	
Apr.	5	17.538	21	152.794	6	5	2	2.9966	
	5	17.560	1	152.859	6	5	2	2.9960	
	5	17.537	1	152.862	5	4	2	2.9960	
	7	17.982	5	152.827	9	4	2	2.9963	
	10	17.050	35	152.871	9	7	3	2.9959	
May	15	20.836	3	152.904	13	3	1	2.9955	
	15	18.979	23	152.922	37	4	1	2.9954	2.99601
	15	19.005	3	152.873	13	3	1	2.9959	
	15	19.005	2	152.866	2	3	3	2.9959	
	15	19.011	2	152.889	20	3	1	2.9957	
	17	19.062	4	152.805	13	3	1	2.9965	
	18	19.017	15	152.856	8	3	1	2.9960	
Oct.	8	20.325	11	152.770	5	2	1	2.9969	2.99683
	9	20.300	18	152.789	61	3	½	2.9967	
Nov.	13	18.390	.....	152.895	.....	1	½	2.9956	
	14	18.354	1	152.825	7	6	3	2.9963	
	14	18.357	2	152.839	3	7	4	2.9962	2.99622
	14	18.360	1	152.829	2	6	3	2.9963	
	14	18.169	2	152.843	4	5	2	2.9962	

TABLE XXVII—Continued.

Date	C	δ_1	Cylinder	δ_2	N	W	v	Mean
1906								
Nov. 14	18.168	1	152.849	4	9	4	2.9961	
15	18.181	2	152.864	4	8	4	2.9959	
15	18.186	2	152.866	3	8	4	2.9959	
15	18.182	1	152.873	3	8	4	2.9959	
15	19.300	2	152.922	3	6	3	2.9954	
15	18.199	3	152.884	3	6	3	2.9957	
15	18.188	1	152.871	3	4	2	2.9959	
16	18.216	1	152.869	5	6	3	2.9959	
16	18.215	1	152.873	5	6	3	2.9959	
17	18.208	2	152.847	5	6	3	2.9961	
17	18.205	1	152.857	4	7	3	2.9960	
17	18.205	2	152.869	5	8	4	2.9959	
17	18.207	1	152.855	6	6	3	2.9960	
17	18.197	2	152.840	9	8	3	2.9962	
17	18.195	4	152.854	5	4	2	2.9960	
Dec. 4	16.799	2	152.793	2	6	3	2.9966	
4	16.797	2	152.792	5	6	3	2.9966	
4	16.806	8	152.806	3	4	2	2.9965	2.99616
4	16.855	4	152.806	10	8	3	2.9965	
5	18.117	2	152.814	2	4	2	2.9964	
6	18.563	3	152.792	2	6	3	2.9966	
6	18.565	2	152.792	6	6	3	2.9966	
6	18.562	1	152.797	4	8	4	2.9966	
6	18.559	2	152.818	8	6	3	2.9964	
7	18.534	10	152.833	19	6	1	2.9962	
7	18.528	10	152.853	9	6	2	2.9961	
7	18.494	1	152.836	2	4	2	2.9962	
1907								
Jan. 5	17.799	1	152.790	4	8	4	2.9967	
5	17.768	1	152.790	4	6	3	2.9967	
5	17.770	2	152.788	4	6	3	2.9967	
Feb. 4	20.619	0	152.878	6	7	3	2.9958	
4	20.619	0	152.890	3	6	3	2.9957	
4	20.617	2	152.884	7	8	4	2.9957	
4	20.612	1	*76.745	2	6	3	2.9957	
4	20.604	1	*76.752	2	6	3	2.9957	2.99570
4	20.608	1	*76.758	2	6	3	2.9957	

TABLE XXVII—Continued.

Date	C	δ_1	Cylinder	δ_2	N	W	v	Mean
1907								
Feb. 4	20.615	1	152.879	3	6	3	2.9958	
5	20.627	1	152.835	2	6	3	2.9962	
5	20.627	2	152.854	4	6	3	2.9960	
5	20.629	1	152.876	4	6	3	2.9958	
5	20.632	1	152.885	8	6	2	2.9957	
5	20.639	2	152.843	2	6	3	2.9962	
5	20.631	2	152.851	4	6	3	2.9961	2.99596
7	20.409	3	152.854	4	8	4	2.9960	
7	20.409	1	152.864	3	6	3	2.9959	
7	20.411	1	152.867	3	6	3	2.9959	
7	20.419	1	152.864	3	6	3	2.9959	
7	20.421	1	152.864	3	7	3	2.9959	

* The middle section of the condenser consisted of section three only.

The average deviation of the results in the eighth column of the table from this mean is

$$0.00030 \times 10^{10}, \text{ or } 10 \text{ parts in } 100{,}000$$

The average value of δ_2 for the remainder of the observations in this table is 0.0046 micro-microfarads. The weighted mean of these 329 independent determinations is

$$v = 2.99609 \times 10^{10}$$

and the average deviation from this mean is

$$0.00027 \times 10^{10}, \text{ or } 9 \text{ parts in } 100{,}000$$

The mean of all the values as they stand is $v = 2.99614 \times 10^{10}$; this, however, gives undue weight to the results obtained on those days on which several series of observations were taken. These series are all independent so far as the electromagnetic measurements go, but are mere duplicates so far as the electrostatic capacity is concerned. Hence it seems more fair to give to the mean result of any one day the mean weight attached to the separate series; thus from the daily means we find

$$v = 2.99624 \times 10^{10}$$

The average variation of the daily means from this is

$$0.00027 \times 10^{10}, \text{ or } 9 \text{ parts in } 100{,}000$$

56. RESULTS BY THE PLATE CONDENSERS.

As already stated the importance of the correct adjustment of the plates to parallelism, as well as the importance of the lower plate being truly plane, was not recognized until the conclusion of the work, hence all results obtained are in error from these causes. Also, until near the end of the work (October, 1906), the importance of the plate being strictly in the plane of the guard ring was not recognized. As a consequence the adjustment was but approximate and the error due to this cause was large. Later the adjustment was made with more care, but it was not possible to make the error from this cause negligible.

Consequently, all results obtained by use of the plates have to be corrected for these two sources of error.

By dividing the results into a series of groups such that each group covers a period over which the adjustment of the plates was not knowingly altered, and by assuming that the adjustment (so far as the displacement of the plate from the plane of the guard ring, and the relative inclination of the plates are concerned) remained unchanged throughout each group, we can determine the errors in adjustment which must be assumed in order to bring into accord with an assumed value of **v** and with one another the results obtained at different distances between the plates.

Proceeding in this manner we have found that all the results obtained can be made concordant with one another and with the value of **v** given by the other condensers by the assumption of very probable errors in these two adjustments. Until April 17, 1906, there was no attempt made to maintain the temperature of the condenser constant. As a consequence, the results before this date are more variable than the others, and it seems scarcely worth while to present them in detail since the essential features of the uncorrected results and of the corrections are sufficiently shown by the later and more reliable work. We may, however, state that the 66 determinations made before this date can be made to give a mean of 2.99627 by assuming displacements of the plate not exceeding 68 μ, and inclinations not exceeding 4′ 42″. The capacity was varied in the ratio of 1 to 15, and the average deviation from the mean given above is but 6 in 100,000.

The other results are set forth in Tables XXVIII and XXIX.

TABLE XXVIII.

Results by Plate Condenser. Plate A.

(Temperature controlled.)

Date	Distance between the plates in mm							
	1.64	2.64	3.64	4.64	5.64	6.64	7.64	9.64
1906								
Apr. 17	2.9971	2.9972						
25		2.9966						
26		2.9969						
28		2.9967						
28	2.9976	2.9965						
30		2.9971						
May 1		2.9969						
1		2.9963						
5	2.9972	2.9967	2.9966					
9			2.9954	2.9951	*2.9939			
9			2.9964		2.9946	2.9952	2.9943	2.9928
9					2.9957		2.9948	
11		2.9967	2.9967		2.9957		2.9947	2.9937
11			2.9960		2.9951		2.9946	2.9946
Mean	2.99730	2.99676	2.99622	2.9951	2.99528	2.9952	2.9946	2.9937
Correction	−0.00143	−0.00041	+0.00028	+0.00077	+0.00115	+0.0014	+0.0017	+0.0020
v	2.99587	2.99635	2.99650	2.99587	2.99643	2.9966	2.9963	2.9957

Mean of all $v = 2.99626 \times 10^{10}$

$p_1 = 0.64$ mm; $q - p = 66.5\ \mu$; $\delta = 94.5\ \mu$; $\theta = 3'\ 15''$

* Omitted from mean.

In Table XXVIII are contained the results obtained by the use of Plate A, when the temperature of the condenser was maintained constant. These fall into a single group and but one initial distance, $p_1 = 0.64$ mm, was employed. Hence the uncorrected values have been tabulated and averaged and the corrections are applied to the means. Though the uncorrected means range from 2.9973 to 2.9937, it is necessary to assume a displacement of but 66.5 μ and an inclination of but $3'\ 15''$ in order to make them range but from 2.9966 to 2.9957, and now the high and low values alternate while the uncorrected values steadily decreased as the capacity was reduced. The mean of all these results is 2.99626×10^{10}.

TABLE XXIX.

Results by Plate Condenser.—Plate B.

Date	p_1 mm	$p-q$ μ	δ μ	θ	1			2			3			4			5			10			15			20			25		
					O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V
1906.																														
May 11	0.68	170	161	5′ 32″				58	+2	60	45	+18	63																	
25								56	+2	58	46	+18	64																	
25								69	+2	71																				
29								47	+2	*49	78	+18	*96																	
29								65	+2	*67	72	+18	*100																	
29											58	+18	*76																	
Oct. 3	0.66	2.7	58	2′ 0″	71	−8	63							69	−3	66														
8					72	−8	64							64	−3	61														
1907.																														
Jan. 5	0.59	5.3	37.5	1′ 17″	67	−4	63				56	−1	55																	
8																	63	+0	63	69	+2	71								
10								65	−2	63							61	+0	61	63	+2	65	53	+3	56	31	+4	*35	45	+4	49
10																							75	+3	78						
18																										72	+4	76	70	+4	74
18																													58	+4	62
24	0.58	10.6	73.0	2′ 31″	85	−17	68										66	−2	64	63	+2	65	60	+4	64	48	+5	53	41	+6	47
24																										43	+5	48	41	+6	47
25					74	−17	57										65	−2	63	66	+2	68									
28	0.60	13.3	47.4	1′ 38″	68	−5	63										59	+2	61	50	−5	55				40	+8	48			
Feb. 1					78	−5	73										62	+2	64	56	+5	61	41	+7	48	52	+8	60			
1					57	−5	52													57	+5	62									
1																				61	+5	66									
Mean corrected value					2.99628			2.99630			2.99607			2.99635			2.99626			2.99641			2.99615			2.99570			2.99558		

Mean omitting three smallest capacities is $v=2.99630\times10^{10}$.

* Omitted from mean.

The results obtained by the use of Plate B are contained in Table XXIX. The temperature throughout each day's work was maintained constant. In the first column is given the date, then follows p_1, in mm, the initial distance between the plates as calculated from the measured electromagnetic capacity and the assumed value $\mathbf{v} = 2.9963 \times 10^{10}$. In the third column is given, in microns, the distance q–p by which the plane of the plate is below the plane of the ring; in the next column is given, in microns, the maximum distance δ of the rim of the collector plate from the imaginary plane through its center and parallel to the movable plate; at least, this would be the value of δ were the lower plate plane; as it is somewhat cylindrical in form the value of δ is greater than that given by reference to the equivalent mean plane of the lower plate. In the following column is given the effective inclination of the plates, $\theta = \tan^{-1} \frac{\delta}{R}$ The numbers at the heads of the sixth and succeeding columns denote the approximate distance, in mm, between the plates. Each of these columns is divided into three parts; in the first (O) is contained the fourth and fifth significant figures of the values of $\mathbf{v}$ as deduced from the observations without applying the correction for the inclination and the displacement of the collector plate; the middle portion contains the correction (C) due to these causes, and the third portion contains the fourth and fifth significant figures of the corrected values of $\mathbf{v}$. The first three significant figures of $\mathbf{v}$ are in every case 299. At the bottom of the table are given the averages of the corrected values corresponding to each distance between the plates, values marked with an asterisk being omitted. These means agree very well with one another, though the capacities measured varied in the ratio of 1 to 10. Their mean is $\mathbf{v} = 2.99630 \times 10^{10}$.

From this table it is easy to see the advantage of the pivoted lever (page 454) in the adjustment of the plates. In May the plate was adjusted by the use of the eye and fingers only, and the displacement of the plate was 170 microns, and the inclination was 5′ 32″. On October 3, January 5, 24, and 28, the plate was displaced and readjusted by the use of the lever. Under these conditions the displacement varied but from 3 to 13 microns, and the inclination from 1′ 17″ to 2′ 31″.

That the values obtained from the smaller capacities are somewhat erratic is not surprising. If we assume that the uncertainty in the electromagnetic measurements is 0.01 micro-microfarad (this is but two-thirds of the average deviation of the measured capacity of the cylindrical condenser from its mean on the same day, page 591), then the uncertainty introduced by this cause alone is as follows:

Distance between plates	1	5	10	15	20	25	30 mm.
Uncertainty in v	0.5	2.7	5.4	8.4	10.8	13.5	15.3×10^{6}.

That is, at 20 mm distance between the plates the values found for v may vary between 2.9974 and 2.9952, even when the uncertainty in the electromagnetic measurements amounts to but 0.01 micro-microfarad, or a capacity of 90 microns.

Though the results obtained from the plates are of no value in establishing the value of v, they are not at all discordant with the results obtained by the use of the other condensers. Realizing now the importance of these adjustments we shall take pains when we return to this work to reduce these errors to a minimum, and trust then to be able to obtain, by the use of plates, results of positive value.

57. DISCUSSION OF RESULTS.

From what has been already stated it is evident that the results obtained by the use of the plates and of the cylinders without guards are of a negative value—they do not contradict the conclusions that may be drawn from the results obtained by the use of the other condensers.

The results obtained by the use of the spheres and of the guard cylinders and contained in Tables XXI, XXII, and XXVII are shown graphically in Fig. 44. Each value of v given in these tables is represented by a dot, of which the size denotes the weight attached to this value, and the ordinate gives the value of v corresponding to it. The dots are in order of the corresponding determinations, from left to right. The dotted lines represent the values of v corresponding to the weighted means of the various groups.

These results are also summarized in Table XXX, where Δ is the average variation of the members of each group from their mean,

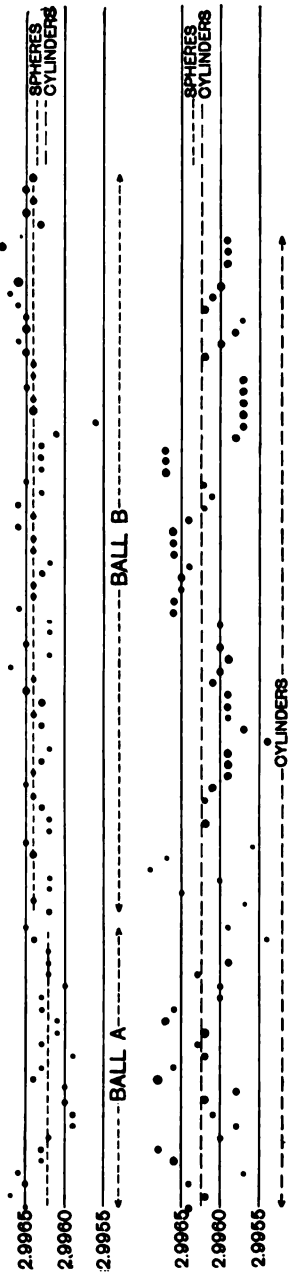

Fig. 44.—*Graphical Summary of Results.*

Δ_1 is the corresponding average variation from the weighted mean of all (2.99629), N is the number of independent determinations.

TABLE XXX.

Date.	Condenser.	V	Δ	N	Δ_1
1905	A	2.99621×10^{10}	0.00018×10^{10}	64	0.00018×10^{10}
1905	B	2.99637 "	0.00011 "	146	0.00013 "
1906	B	2.99659 "	0.00017 "	89	0.00030 "
1906	Cylinders	2.99624 "	0.00027 "	603	0.00028 "
Weighted Mean		2.99629 "	Mean		0.00022 "

The abnormally high value obtained by the use of ball B in 1906 has already been noticed. Being based upon the observations of but five days, it is not to be given undue importance, but nevertheless it does indicate a possible outstanding uncertainty of some kind.

From the concordance of the other observations we believe that the mean value (obtained from a series of observations extending over several months) of the square root of the ratio of the electrostatic to the electromagnetic capacity of an air condenser is

$$v = 2.9963 \times 10^{10} \left[\frac{\text{cm Int}\Omega}{\text{sec}}\right]^{\frac{1}{2}}$$

with an uncertainty of not more than 1 in 10,000.

Since the mean temperature at which the work was done was about 20°.0, this gives us for the value reduced to vacuo (assuming the dielectric constant of air at 20°.0 C and 760 mm pressure to be 1.00055)

$$v_0 = 2.9971 \times 10^{10}$$

We have given above a very full discussion of the sources of error in the determination of capacity in electrostatic and in electromagnetic measure, partly for the sake of the present work and partly because we think it desirable that the accuracy of the work be carried further in the near future. As we have already stated, the accuracy we expected to attain when the work was taken up nearly three years ago was considerably short of what we believe we have actually accomplished. We then regarded errors negligible